

What is the Weibull distribution?

... why is it even a thing I need to care about?



Workbook

Presented by

Dr Chris Jackson

Reliability engineer



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Who is your teacher? ... Dr Chris Jackson

Dr Jackson holds a Ph.D. and MSc in Reliability Engineering from the University of Maryland's Center of Risk and Reliability. He also holds a BE in Mechanical Engineering, a Masters of Military Science from the Australian National University (ANU) and has substantial experience in the field of leadership, management, risk, reliability and maintainability. He is a Certified Reliability Engineer (CRE) through the American Society of Quality (ASQ) and a Chartered Professional Engineer (CPEng) through Engineers Australia.

Dr Jackson is the cofounder of is4 online learning, was the inaugural director of the University of California, Los Angeles (UCLA's) Center of Reliability and Resilience Engineering and the founder of its Center for the Safety and Reliability of Autonomous Systems (SARAS). He has had an extensive career as a Senior Reliability Engineer in the Australian Defence Force, and is the Director of Acuitas Reliability Pty Ltd.

He has supported many complex and material systems develop their reliability performance and assisted in providing reliability management frameworks that support business outcomes (that is, make more money). He has been a reliability management consultant to many companies across industries ranging from medical devices to small satellites. He has also led initiatives in complementary fields such as systems engineering and performance-based management frameworks (PBMFs). He is the author of two reliability engineering books, co-author of another, and has authored several journal articles and conference papers.

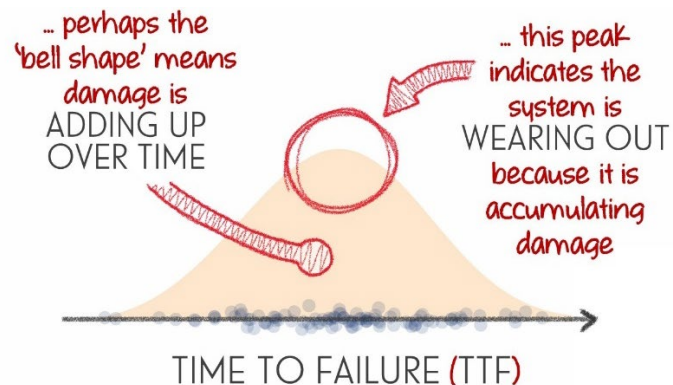


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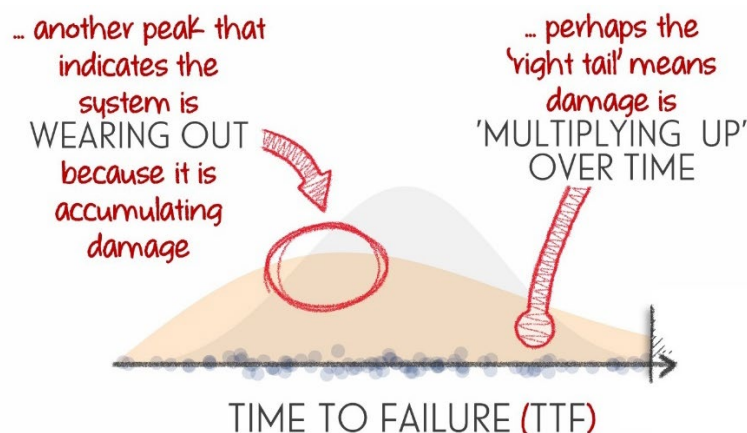
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The Weibull distribution

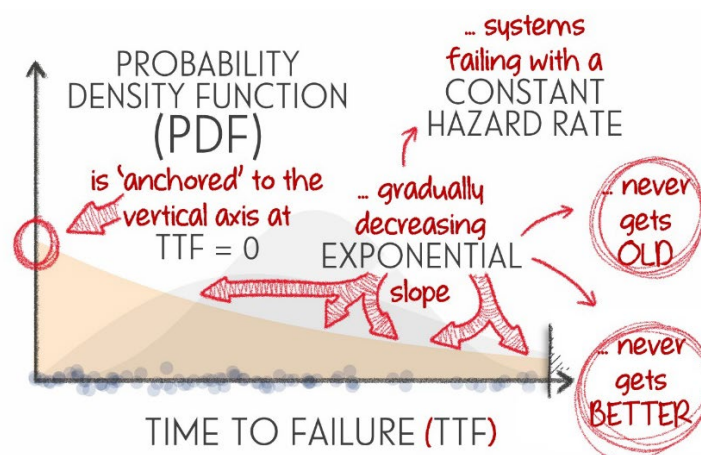
The **Weibull distribution** is talked about a *lot* in **reliability engineering**. And there is a really good reason why. But first, some revision. We know that the **probability density function (PDF)** defines a curve that defines the relative likelihood of a **random variable** having a certain value. That means, random variable values are more likely where the **PDF curve** is the highest (as in the illustration below). And the shape of that curve can tell us a lot about how failure is occurring ... if the random variable in question is **time to failure (TTF)**.



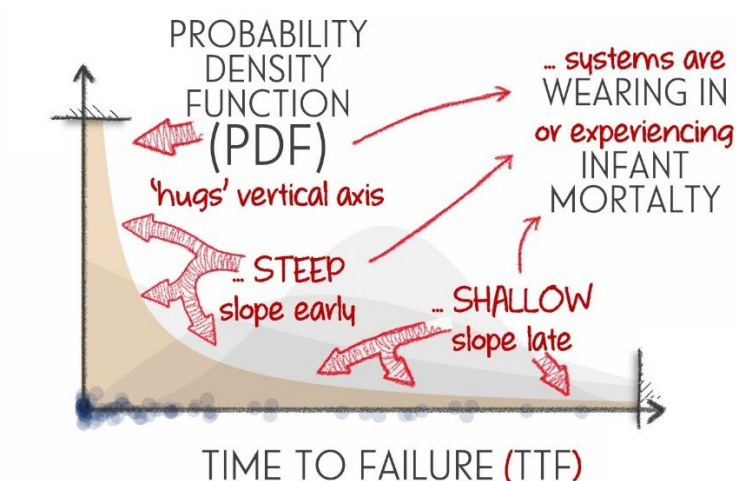
If we become really proficient, at this we can not only tell if our system is wearing out, but HOW it is wearing out. Another PDF curve with a 'wear-out' peak is illustrated below.



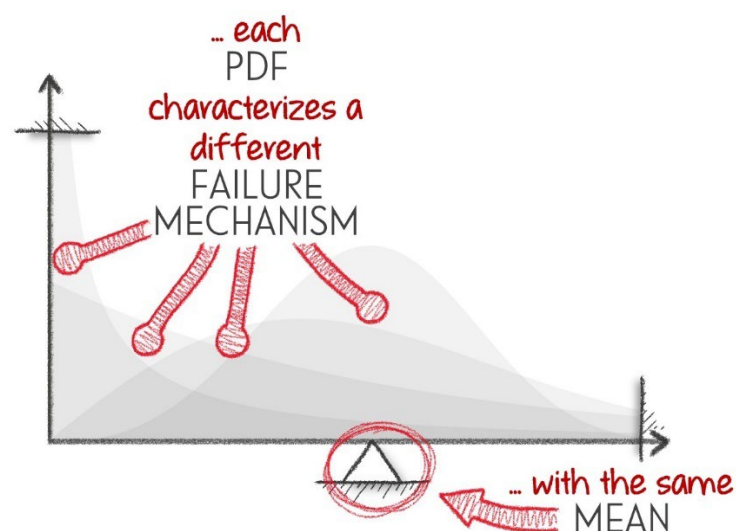
Then there are **PDF** curves with this shape ...



And **PDF curves** with this shape ...



All the **PDF curves** in the previous illustrations (each one modelling a different **failure mechanism**) were all specific examples of the **Weibull distribution**. And ... they all had the same **mean** or **mean time between failure (MTBF)**. This further illustrates that just understanding the mean of the **random process** that is **failure** reveals very, very little about the **reliability characteristics**.



The ability to model lots of different, well known failure mechanisms is primarily why the **Weibull distribution** is so popular. If you don't know what your data 'says' about the nature of **failure**, conducting what we call a '**Weibull analysis**' can not only find (perhaps) a really good **TTF** model for your product, system or service, but it can tell you where you need to look in order to improve **reliability**.

... Weibull distribution PARAMETERS

The parameters of the **Weibull distribution** are talked about ... a lot. Beyond helping us define the shape of the **PDF** and **reliability curves**, they tell us a lot about the nature of **failure**.

shape parameter - β

The **shape parameter** is represented by the lowercase Greek letter beta (β). The **shape parameter** tells us if our product, system or service is wearing in, wearing out, or has a constant **hazard rate**

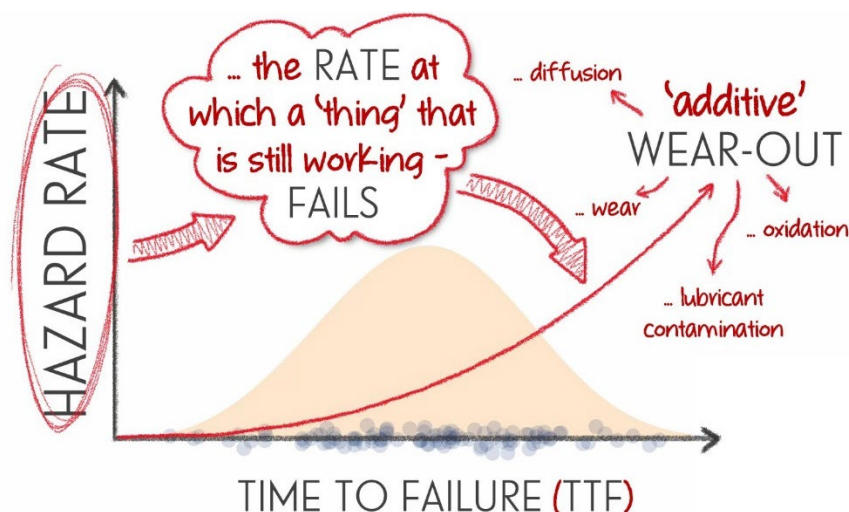
if the shape parameter (β) is ... then the product, system or service is/has ...	
less than 1	wearing in
equal to 1	a constant hazard rate
greater than 1	wearing out

scale parameter - η

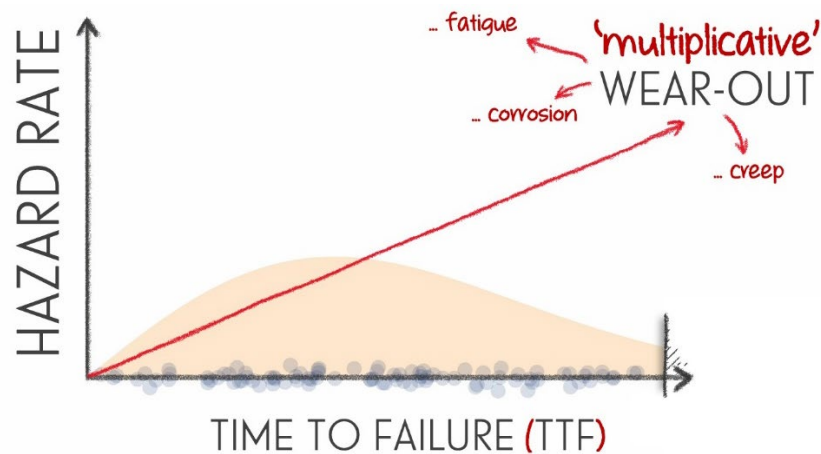
The **scale parameter** is represented by the lowercase Greek letter eta (η). The **scale parameter** tells represents an indicative size of the **Weibull random variable** in question. The **scale parameter** is the 63rd percentile (which is just a value that makes it easy for us to express the **probability distribution functions**).

... some examples

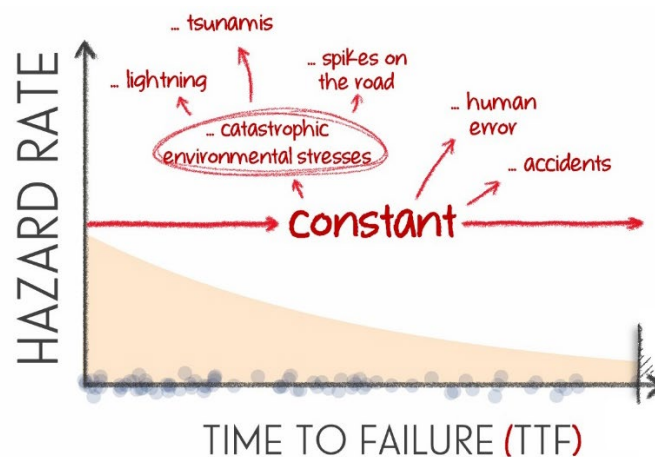
Because the **Weibull distribution** can model lots of different **failure mechanisms**, it (very understandably) it can look very similar to lots of other **probability distribution**. For example, the **PDF** and **hazard rate curves** for a **Weibull distribution** with a **shape parameter of 3.5** are illustrated below. This closely mimics the **normal distribution**.



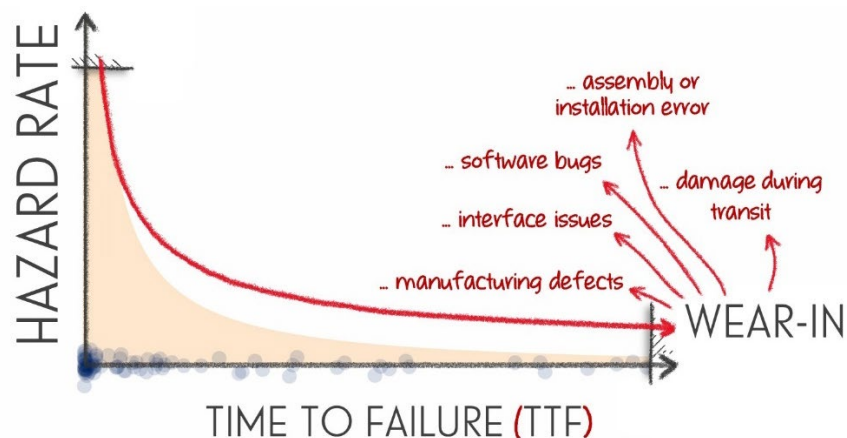
The **PDF** and **hazard rate curves** for a **Weibull distribution** with a **shape parameter of 2** are illustrated below. This closely mimics some **lognormal distributions**.



The **PDF** and **hazard rate curves** for a **Weibull distribution** with a **shape parameter of 1** are illustrated below. This is the **exponential distribution**.

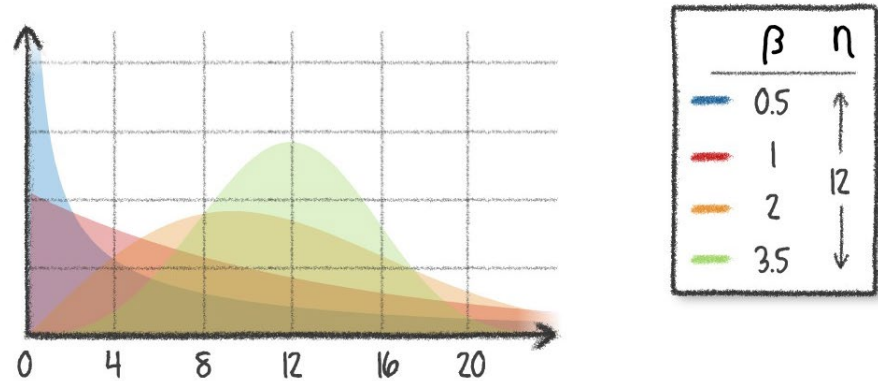


And finally, the **PDF** and **hazard rate curves** for a **Weibull distribution** with a **shape parameter of 0.4** are illustrated below. This does not mimic a specific distribution, but has been found to accurately model wear-in failure mechanisms.



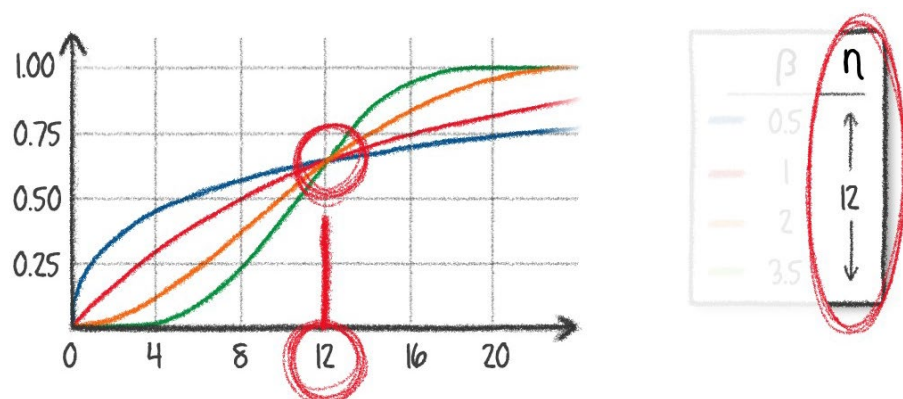
PROBABILITY DENSITY FUNCTION (PDF)

The equations and formulae for the **PDF** and other functions and statistics based on the **Weibull distribution** (including the **mean**) are at the end of this chapter. The most important thing to understand when dealing with **PDFs** is knowing what they represent in terms of failure. Four example **Weibull distribution PDF curves** are illustrated below. They all have different **shape parameters**, but identical **scale parameters** (12).



CUMULATIVE DISTRIBUTION FUNCTION (CDF)

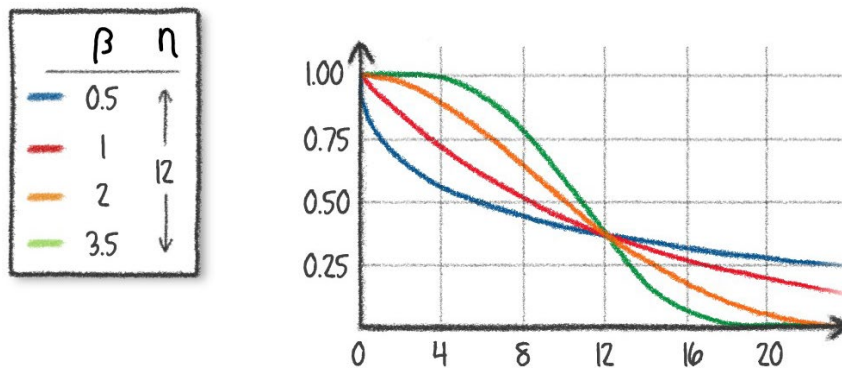
Four example **Weibull distribution CDF curves** are illustrated below. Each **CDF curve** corresponds to one of the **PDF curves** above. You can see here that the **Weibull distribution CDF** (like all other **CDFs**) starts at 0 and approaches 1 (or 100 %) as **TTF** increases.



You can also see that regardless of the **shape parameter** (β), all **CDF curves** above pass through the same point. This point happens to occur when the **TTF** is equal to the **scale parameter** (η). This is a very useful characteristic that can be exploited when we are trying to identify **parameters** for a given failure process. We will explore that later.

RELIABILITY FUNCTION

Four example **Weibull distribution reliability curves** are illustrated below. Each **reliability curve** corresponds to one of the **PDF and CDF curves** above. You can see here that the **Weibull distribution reliability curve** (like all other **reliability curves**) starts at 1 (or 100 %) and approaches 0 as **TTF** increases.



You can see that again, all these curves pass through the same point when the TTF is equal to the scale parameter (η). There is a reason for this. The Weibull distribution reliability function is ...

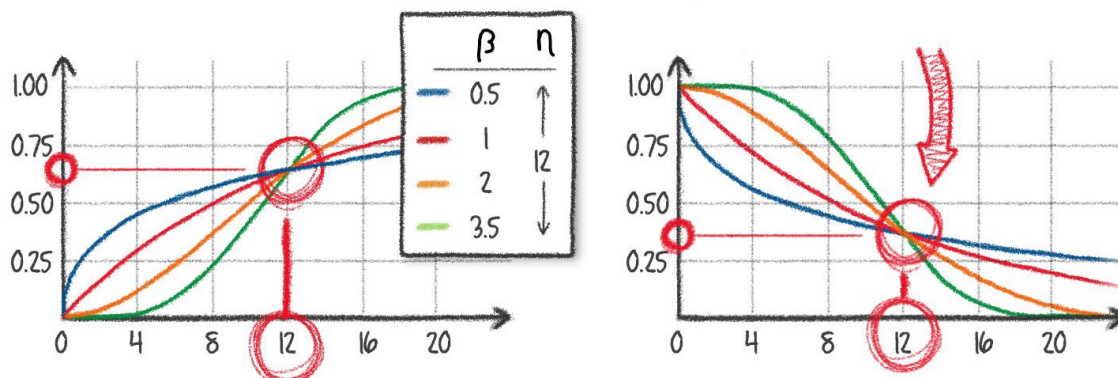
$$R(t) = e^{-\left(\frac{t}{\eta}\right)^\beta}$$

Now we can use this equation to understand 'what is happening' when our system has reached its characteristic life ...

$$R(\eta) = e^{-\left(\frac{\eta}{\eta}\right)^\beta} = e^{-(1)^\beta} = 36.788 \dots \%$$

... when the system has reached its CHARACTERISTIC LIFE ($t = \eta$) ...

Which is why we always see this ...



HAZARD RATE

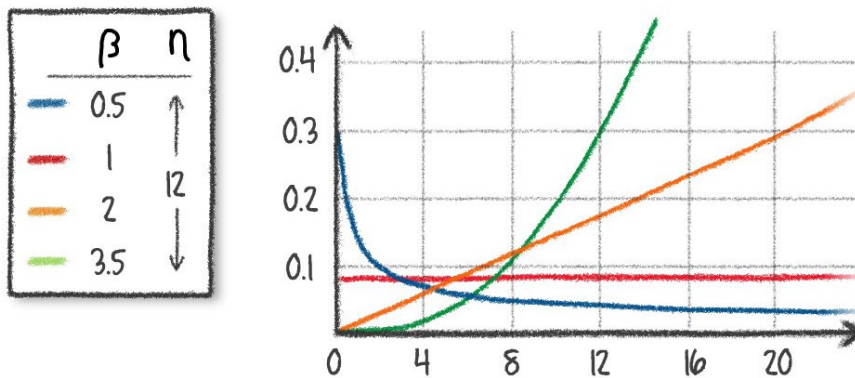
One way of looking at the **hazard rate** is by seeing it as the rate at which working (non-failed) systems are **failing**.

$$\frac{f(t)}{R(t)}$$

... the RATE at which a 'thing' that is still working - FAILS

PROBABILITY DENSITY FUNCTION (PDF)
RELIABILITY FUNCTION

Four example **Weibull distribution hazard rate curves** are illustrated below. Each **hazard rate curve** corresponds to one of the **PDF, CDF and reliability curves** above.



This is where you can see the way in which the **Weibull distribution** can model all sorts of different types of **failures**, with it possible to have decreasing, constant and increasing **hazard rates**. But ... the **Weibull distribution** cannot model products, systems or services that have a little bit of each. You need to create a **system reliability model** for that.


MEAN

The **Weibull distribution mean** is a little complicated to calculate.


WEIBULL DISTRIBUTION MEAN

$$\mu = \eta \Gamma \left(1 + \frac{1}{\beta} \right)$$

GAMMA FUNCTION $\Gamma(x) = \int_0^{\infty} z^{x-1} e^{-z} dz$



This is obviously difficult (impossible) to calculate by hand. Microsoft Excel formulae that can help you find the **Weibull mean** are ...

 $= [\eta] * \text{GAMMA}(1 + (1/[\beta]))$
 $= [\eta] * \text{EXP}(\text{GAMMALN}(1 + (1/[\beta])))$

EXCEL[®]
2003 &
earlier ...

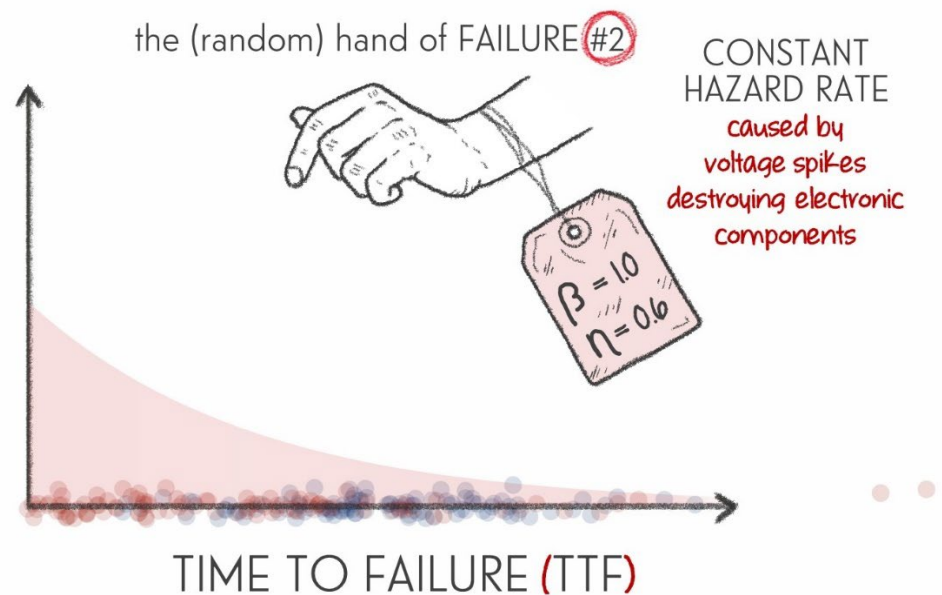


... a quick look at 'WEIBULL ANALYSIS'

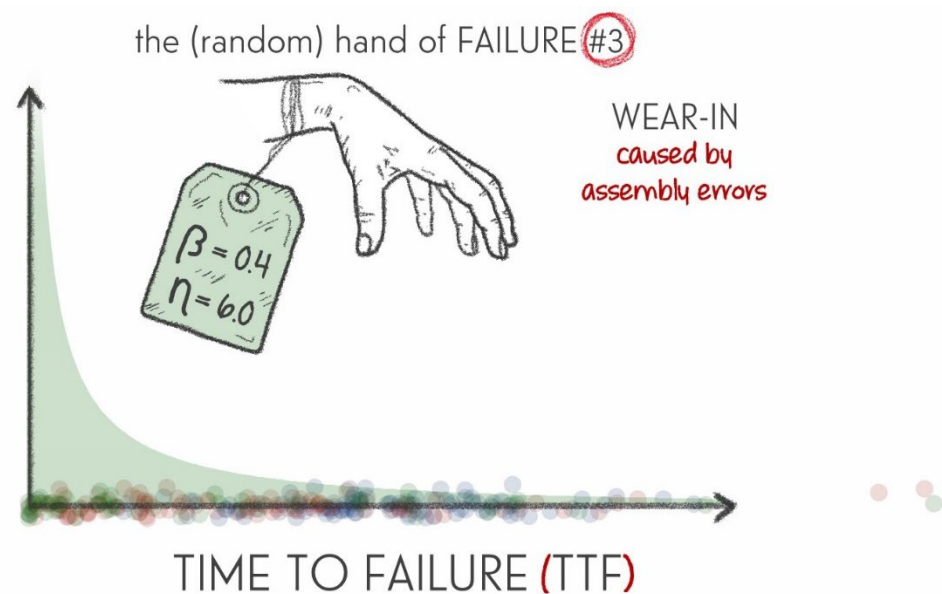
Let's go back to our 'random hand of failure.' It can represent a **reliability test**, actual use, historical data or anything else where your product, system or service is allowed to operate until **failure**. Let's say we conduct a lifetime test on a **failure process** described by the **Weibull distribution** with a **shape parameter** of 3.5 and a **scale parameter** of 1.0.



Let's say we conduct another lifetime test on a **failure process** described by the **Weibull distribution** with a **shape parameter** of 1.0 and a **scale parameter** of 0.6.

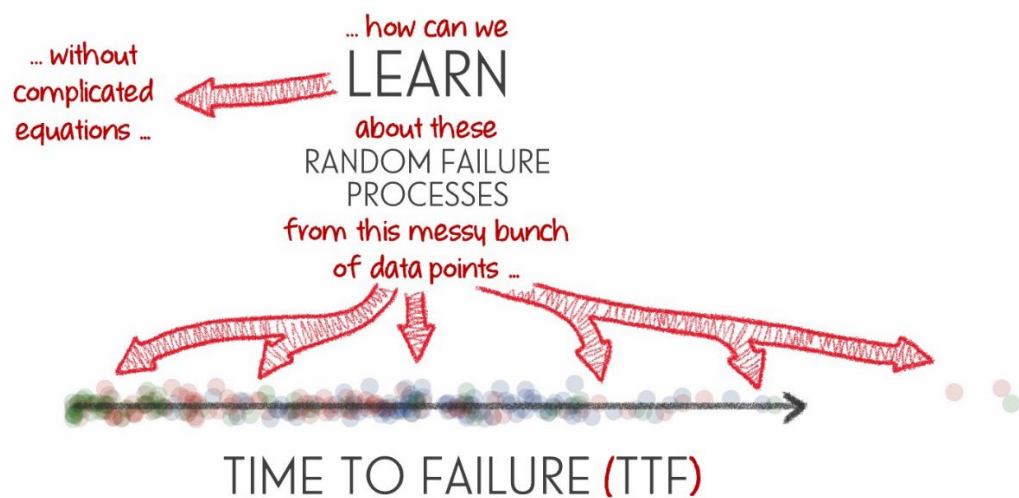


Let's now say we conduct another lifetime test on a **failure process** described by the **Weibull distribution** with a **shape parameter** of 0.4 and a **scale parameter** of 6.0.

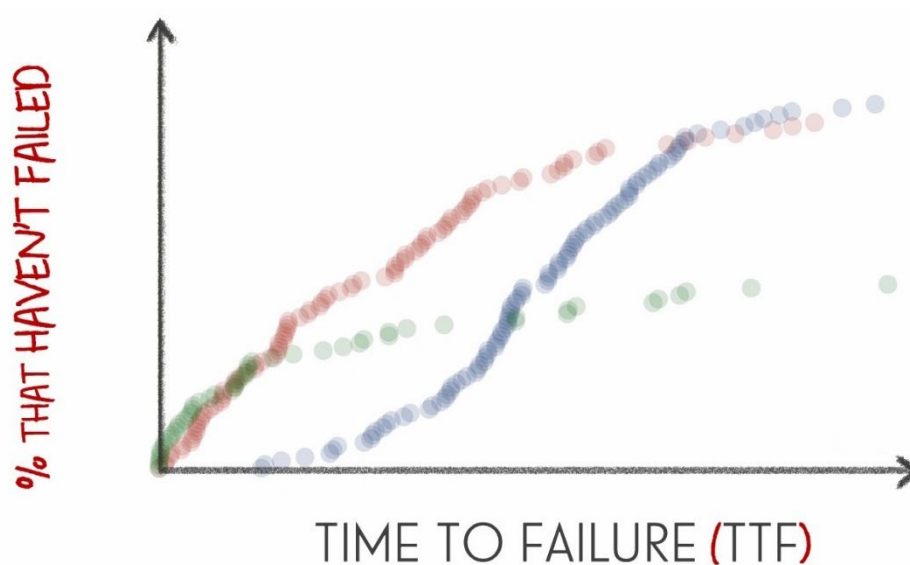


We can see that each process will result in test data which looks different to every other piece of test data.

But ... in the real world we don't know what those 'random hands of failure' represent. We don't know that the shape parameter of the first process is 3.5. We don't know that the scale parameter of the third process is 6.0. All we have is a bunch of 'dots' that represent the TTFs we observed.

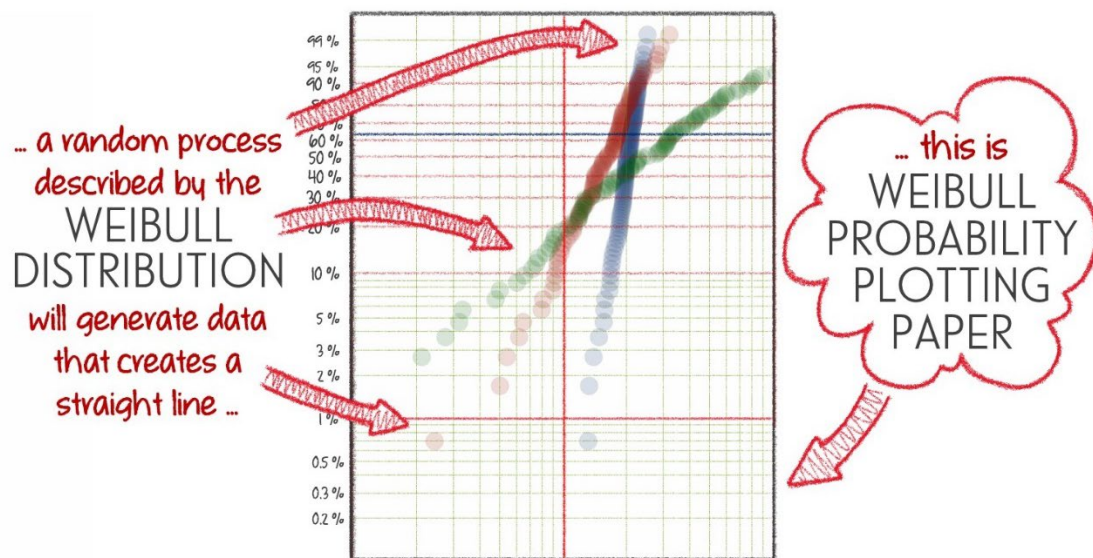


We can get a better idea of what is happening by plotted these data points on a chart where the vertical axis represented the percentage of things that have **failed**. This makes a 'smooth' line that helps us better visualize the 'nature' of **failure**. And we can better 'see' the three separate failure processes below.

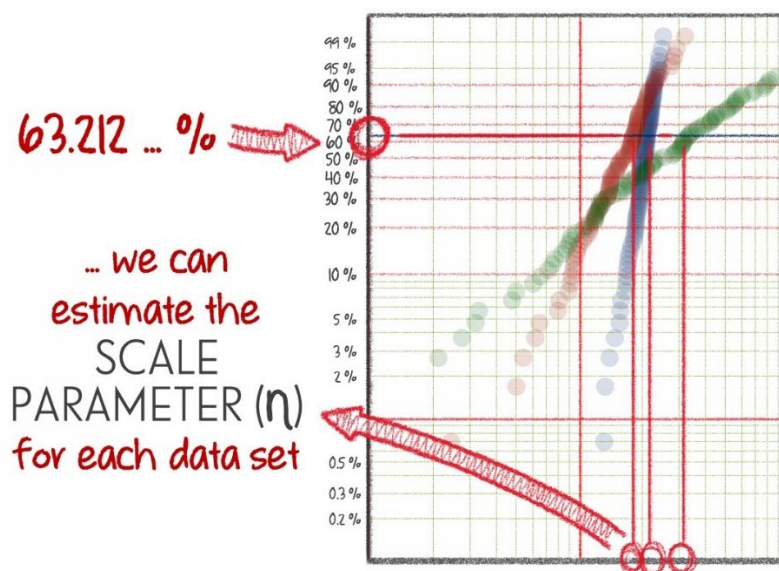


While these lines are 'smooth' ... which usually means it is easier to spot patterns ... it is simply not possible to use it to work out what the hypothetical **Weibull distribution** parameters might be. Or if a Weibull distribution is the right distribution to use.

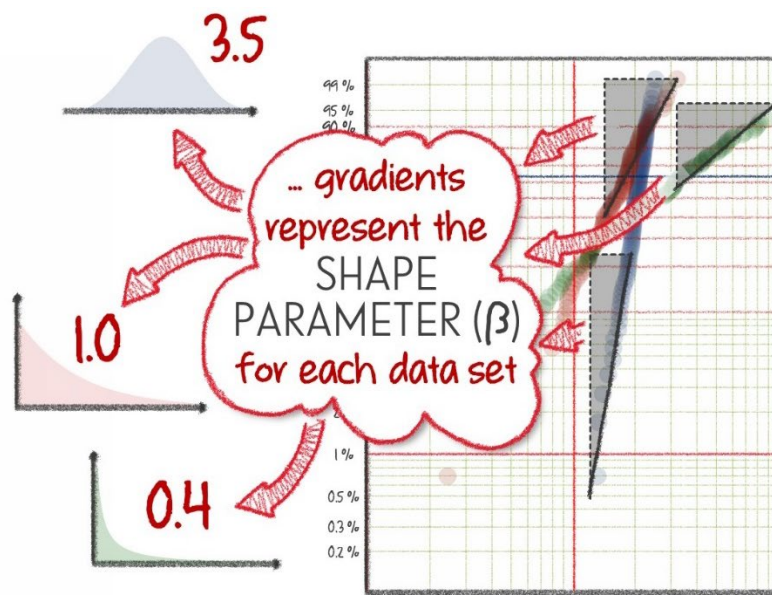
The key to a lot of data analysis activities is to make a 'straight line.' We can manipulate the **Weibull CDF** a little bit to create a 'special' chart that is often referred to as **Weibull probability plotting paper**. The reason we do this is that any data that is described by a **Weibull distribution** will create a straight line on **Weibull probability plotting paper**. The data from the three **failure processes** are illustrated on **Weibull plotting paper below**. Note that the vertical axis represents the **CDF** or **failure probability**.



And that straight line can be used to estimate our **Weibull distribution parameters** directly. We know that because the **CDF** of any **Weibull** will be 63.212 % at the **characteristic life**. So we can find the **scale parameter** from the **Weibull plot** where the **CDF** is 63.212 %.



We can use the slopes of the lines of best fit to estimate the **shape parameter** of each process.

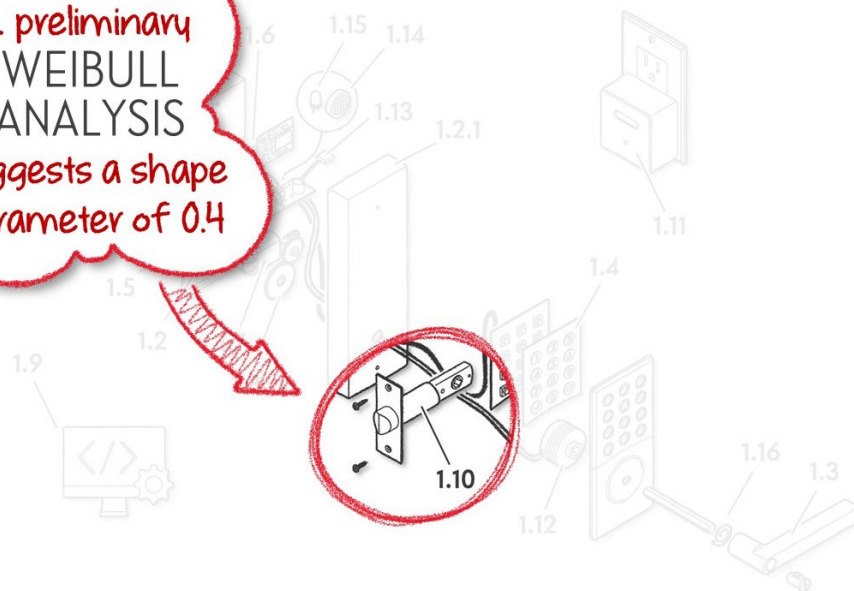


Weibull probability plotting is a lot easier than you might expect. You can do it on specially printed paper or using software to automatically generate **Weibull plots**. Neither approach requires you to use any equations ... but you do need to really understand what every **Weibull plot** represents.

Let's go back to the smart lock.

EXERCISE

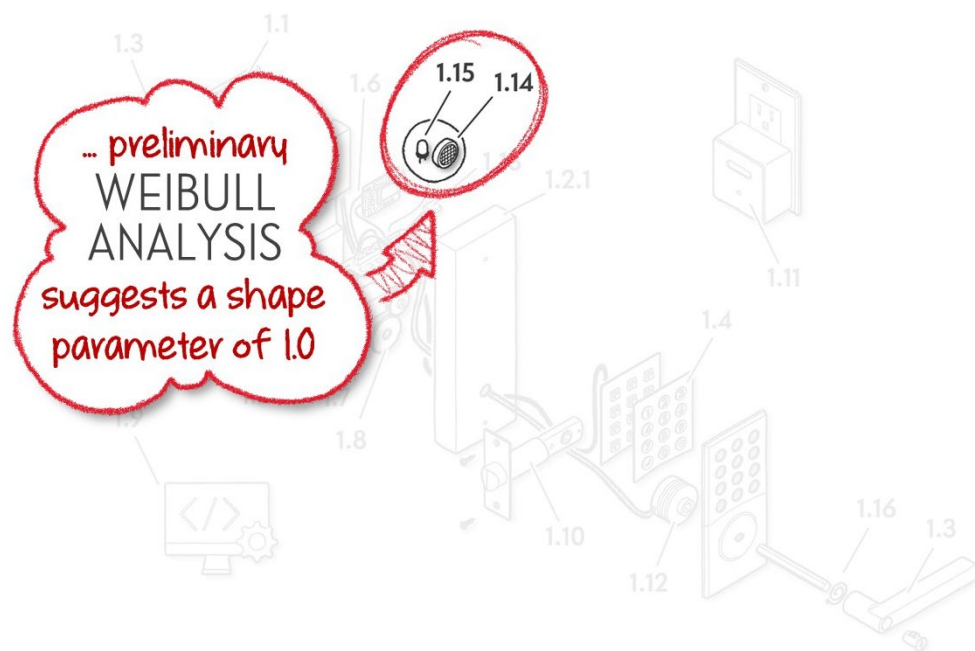
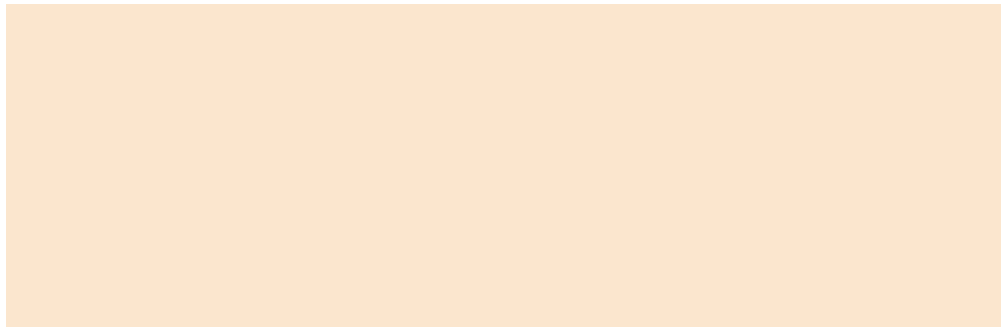
... preliminary
WEIBULL
ANALYSIS
suggests a shape
parameter of 0.4



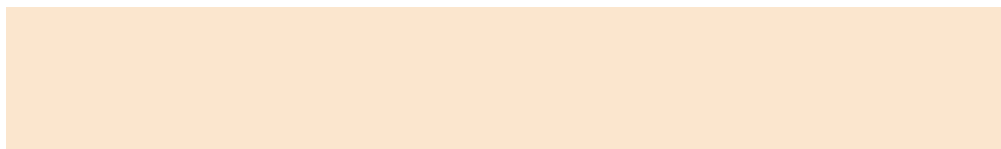
What sort of 'things' might be causing bolt mechanism failure?

EXERCISE

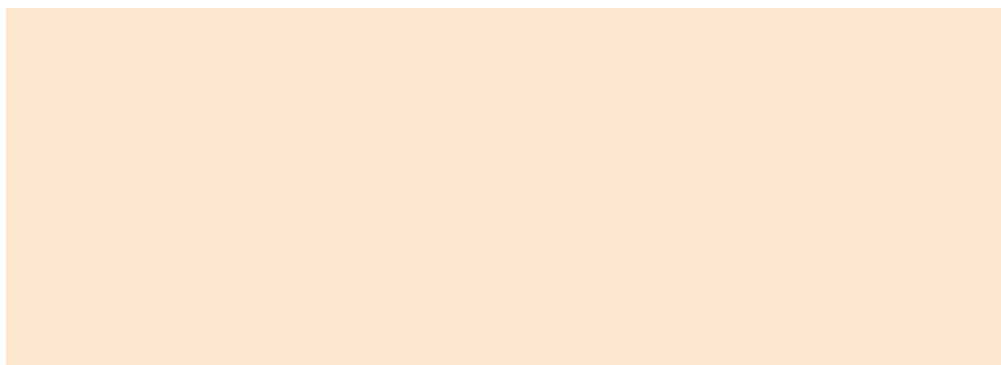
What sorts of things MIGHT we be able to do to improve the reliability of our bolt mechanism?



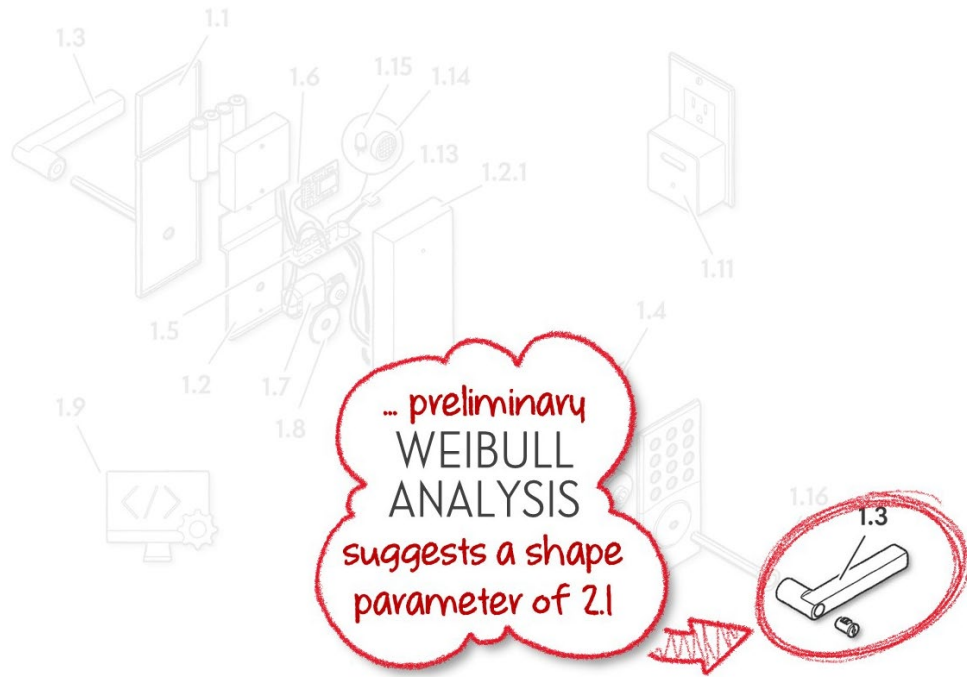
What sort of 'things' might be causing LED or speaker failure?



What sorts of things MIGHT we be able to do to improve the reliability of our LED and speaker?



EXERCISE



What sort of 'things' might be causing handle failure?

What sorts of things MIGHT we be able to do to improve the reliability of our handle?

WEIBULL DISTRIBUTION

Equations and Formulae

Parameters

β – shape parameter

... defines the nature of the random failure process we are modelling
(wear-in, constant hazard rate, wear-out, and different types therein)

η – scale parameter

... defines the magnitude of the random variable (TTF)

Probability Density Function (PDF)

$$f(t) = \frac{\beta t^{\beta-1}}{\eta^\beta} e^{-\left(\frac{t}{\eta}\right)^\beta}$$

Cumulative Distribution Function (CDF)

$$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^\beta}$$

Reliability Function

$$R(t) = e^{-\left(\frac{t}{\eta}\right)^\beta}$$

Hazard rate

$$\lambda(t) = \frac{\beta t^{\beta-1}}{\eta^\beta}$$

Mean

$$\mu = \eta \Gamma\left(1 + \frac{1}{\beta}\right)$$

... where is the gamma function, which is defined as $\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz$

