

Two-Dimensional Non-Hermitian Fermion Systems

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Preface

The goal of this thesis is to reexamine the quantum Hall effect from the perspective of non-hermitian quantum mechanics to see what changes might occur to the computation. This document examines the behavior of two-dimensionally-confined fermions subject to a normal magnetic field without interactions, impurities, or other-such difficulties. While the title describes this thesis as pertaining to fermions in a magnetic field, the only fermion-ness we require is for charged, massive particles that do not occupy the same state.

For at least half a century, conventional quantum mechanics textbooks have tacitly assumed that only hermitian operators (operators equal to their conjugate transpose) may pertain to observables[1][2]. Hermitian observables arise when imposing the self-adjointness of operators over an orthonormal basis, and so they would naturally arise in any practical calculation. The basic point of non-hermitian theories is that a self-adjoint observable operating over a non-normalized basis *is not* necessarily Dirac hermitian[3], despite being physically relevant.

One of the nicest properties of hermitian operators is that they are *guaranteed* to have real eigenvalues, and hence the values of observables would automatically be all-real if all observables were hermitian. However, there are whole classes of non-hermitian operators which may have spectra that are all real, or real for a range of values of non-hermitian parameters. Furthermore, an increasing number of physical systems exhibiting non-hermitian behaviors continue to arise, which theories like PT-quantum mechanics are able to predict with excellent accuracy. Where are systems hermitian, why does a system digress into a non-hermitian regime, and what are the characteristics of a non-hermitian system? By exploring non-hermitian quantum mechanics, we hope that the role of hermiticity will be better understood in addition to perhaps uncovering new physics.

While exploring the depth of this topic, I had hopes to extend my reach into spin-dynamics and the deeper instances of the Hall effect (such as the fractional quantum Hall effect, or spin-momentum locking in topologically non-trivial states of matter). It turns out that describing the dynamics of PT-quantum mechanics even for a straightforward system can be challenging, and so I was forced to limit myself. As it stands, the quantum Hall effect serves as a very basic case from which to springboard into non-hermitian quantum theory, specifically PT-quantum mechanics. In fact, I formulate the simplest toy model I could physically motivate to perturb with a non-hermitian term. The introduction serves to acquaint the reader with the brief, basic, pseudo-quantitative explanation of the behavior of the qHe according to conventional

quantum theory. We re-derive some of the more famous results, such as the Landau levels and energy quantization, and leave out the subtleties. The paper is then free to introduce and explain Non-Hermitian Quantum Mechanics and within the Hall effect and jump into PT-quantum mechanics.

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Abstract

This paper considers the physics of the Hall effect beginning from elementary principles, leading to a non-hermitian perturbation of several quantum Hall toy models. Beginning with the history of the Hall effect, the physics of fermions in a magnetic field is introduced, outlined, and derived. This treatment yields intuition, a grasp of the problem, and a simple, conventional model of a fermion in a magnetic field on a two-dimensional lattice. The precepts of PT-quantum mechanics are then laid out and a non-hermitian Hamiltonian is constructed using a complex momentum perturbation. Finally, results are obtained that suggest a PT-phase transition within the Bloch square lattice model, a novel result for the quantum Hall Hamiltonian.

Dedication

For my mom, my dad, and my brother.

Introduction

Behavior of Fermions in a Magnetic Field

The theory of fermion behavior in a magnetic field is one of the crown jewels of contemporary theoretical physics. Quantum mechanics has predicted a plethora of surprising phenomena decades ahead of experimental confirmations, providing compelling support for conventional quantum theories. Despite these victories, fermion behavior is still one of the most hotly researched topics in physics. Physicists have discovered that fermion behavior is a problem of vast scope. The most important milestones in fermion theory, far from closing off avenues, have begged more questions than they've answered while revealing phenomena beyond the human intuition.

What follows is a qualitative overview of the major phenomena associated with fermions in a strong magnetic field. These explanations contextualize the phenomena referred to in this thesis in a relatable way.

0.1 The Hall Effect

The Hall effect is a well-understood phenomenon that occurs in a current-carrying material subjected to a magnetic field. The phenomenon is totally split into its two regimes– the classical and the quantum mechanical– with each describing vastly different behavior. The classical Hall effect was first discovered in 1879 by Edwin Hall[8]. A full century after the Hall effect was discovered, a hitherto unknown and pronounced behavior called the quantum Hall effect was observed in the laboratory by Klaus Von Klitzing[9]. The seemingly simple and straightforward Hall effect, his paper showed, is not only non-linear, but it also can exhibit incredible quantum effects. In a conductor, charge carriers such as negative electrons or positive charge “holes” are free to move. When a voltage is applied across such a conductor, the charge carriers are forced through the sample, creating current. Classically, a charged particle moving in the presence of a magnetic field experiences a force according to the Lorentz force law,

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}), \quad (1)$$

where e is the charge of a particle, $\vec{v}$ is the velocity vector of the particle and $\vec{B}$ is the magnetic field vector.

Within a thin conducting layer, charged particles move tangent to the magnetic

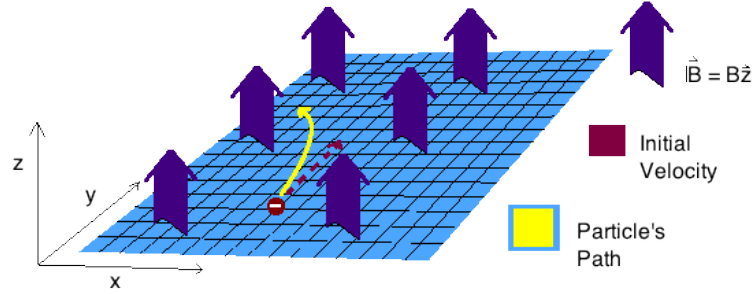


Figure 1: A charged particle confined to a plane is deflected in a normal magnetic field

field at all times where the magnetic field is normal to the surface. With a constant electric field across the surface, positive charges are forced in one direction and negative charges are forced in the opposite direction. According to the Lorentz force law, holes and electrons moving according to the same electric field are forced in the same direction by the same magnetic field. Creating a current moving perpendicular to the magnetic field causes positive and negative charge carriers to be forced apart perpendicular to both $\vec{E}$ and $\vec{B}$, creating a potential. The induced voltage perpendicular to both the electric and magnetic fields is called the Hall Voltage, V_H . The proportionality between voltage, current and magnetic field is a parameter called the Hall resistance R_H [10],

$$R_H = \frac{V_H}{I} = \frac{B_z}{eD}, \quad (2)$$

where e is the charge of the current carrier and D is their density per unit area. R_H changes linearly with B , as we might expect. This linear relation between Hall resistance is exactly what researchers confirmed for nearly a century, before high energy magnetic fields and extremely low temperatures were attainable in the lab.

In 1980, under extreme magnetic fields and close to absolute zero, Klaus Von Klitzing observed R_H behaving like a “staircase function” of B , changing abruptly to some small changes and staying constant for huge changes. His data hinted that R_H is not directly related to the magnetic field. Instead it is given by integer values [9],

$$R_H = \frac{h}{ne^2}. \quad (3)$$

The Hall resistance is an integer fraction multiple of $\hbar/e^2$: no direct dependence on B or I . The Von Klitzing constant,

$$R_K = R_{H1} = R_K = 25,812.807\Omega,$$

being made of fundamental constants, is one of the most precise constants in physics[9]. Von Klitzing’s results show a remarkable coincidence between Hall resistance plateaus and a perpendicular state of 0 resistivity (see Fig.4.). When the Hall resistance is a constant, the resistance perpendicular to the Hall voltage (the current carrying direction) is 0. The zero resistivity state can therefore be switched on and off with relatively small changes in the magnetic field.

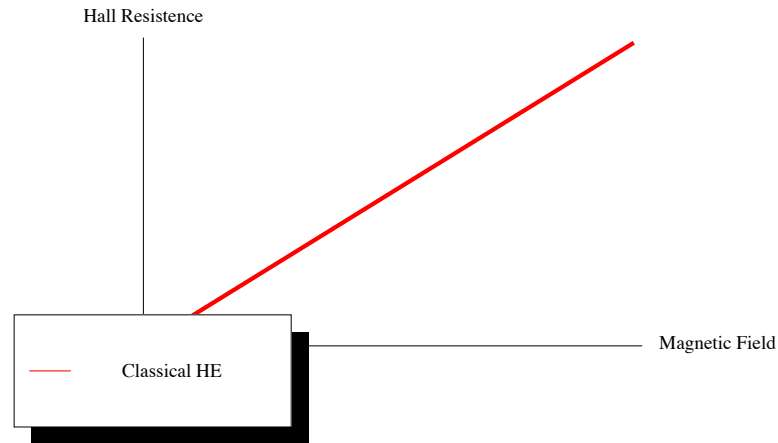


Figure 2: The classical Hall effect (Eq. 2). For this regime, R_H is directly proportional to the magnetic field.

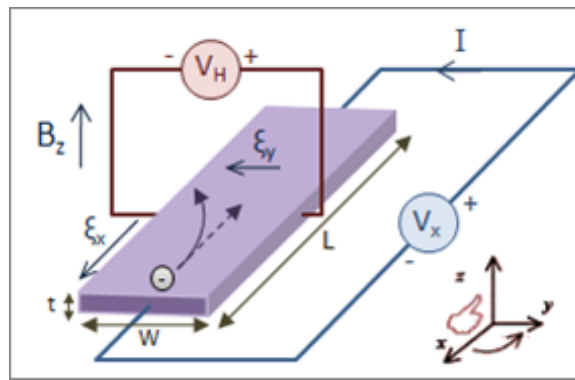


Figure 3: The simple Hall effect, visaged in a simple experimental setup to track the Hall voltage V_H .

Von Klitzing's work opened up considerable research into topological electron states[10]. Among the achievements of his measurement of the QHE is that the Hall Resistance can be used to make a near perfect measurement of the fine structure constant,

$$\alpha = \frac{e^2}{\hbar c}.$$

Later work unearthed the fractional quantum Hall effect[11], the same effect where n is a rational fraction. But how can we reconcile this effect with the classical Hall effect— where is the Lorentz force that explained conductor physics?

Understanding the Quantum Hall effects[12][13]

How can we understand Eq. 3 in terms of our earlier classical result, Eq. 2? How can the Hall resistance not depend on magnetic field or current? As it turns out,

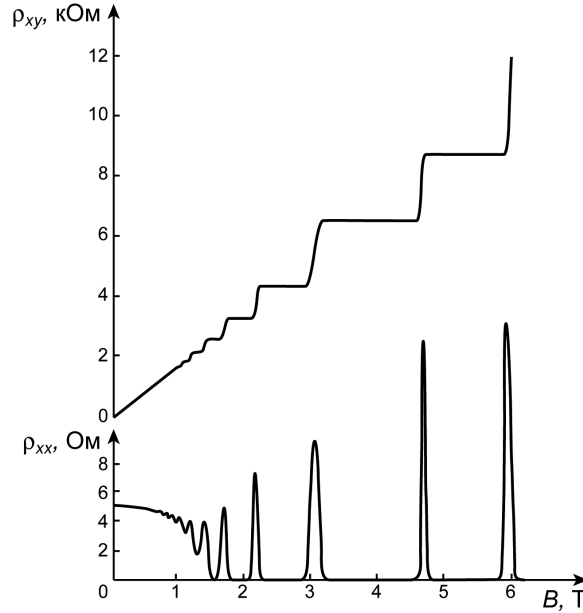


Figure 4: The quantum Hall effect. ρ_{xy} (R_H) is quantized and ρ_{xx} (the resistance along the current) is 0. As the magnetic field increases, the Hall resistance increases as a step function and ρ_{xx} experiences spikes at the lip of the step[5].

Hall quantization has no classical explanation: we require quantum mechanics to provide a mathematical proof. Before deriving it explicitly, a semiclassical, qualitative explanation of the quantum Hall phenomenon follows.

To begin, consider the model of a two-dimensional electron gas, as shown in Fig. 0.1, including some impurities. According to the Pauli exclusion principle, no two fermions occupy the same state. In a low temperature environment, electrons tend to have filled the lowest available states. Wave functions, as we will see later, come in highly degenerate groups called Landau levels—large sets of available states that have the same energy (theoretically speaking). In real life where samples have impurities, Landau levels contain a slim range of energies that a fermion like an electron can occupy. As we will see, the disorder of the system is, in a bizarre twist, the feature of the system that lead to Von Klitzing’s famous observations of macroscopic order.

Within a Landau level, there are low-energy bound states, high-energy bound states, and extended states. For electrons, the low-energy states occur near an impurity of a positive charge, and the high-energy states are localized near impurities of positive charge. Both low and high-energy states are localized in the sample while extended states can move about. The band of mobile electrons in extended states are grouped into what is called the Fermi band, a group of similar energy electron states that have higher energy than the low-energy localized states and the Fermi energy, but lower energy than high-energy localized states. Electrons supplied from the cathode travel through the conductor, and when they encounter a free low-energy state they prefer to occupy it and give off energy in the form of heat. Cooling the system

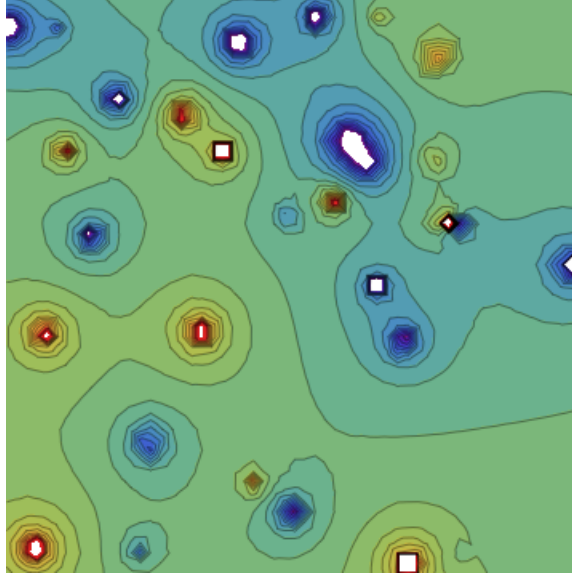


Figure 5: Potential in a 2DEG including impurities. High-potential areas surround impurities of same charge as the charge carriers, and vice versa for low-potential areas. States for charged particles of high and low-energy are localized in the vicinity of like and unlike-charged impurities respectively.

down to very near $0K$ ensures that all of the electrons occupy the lowest energy states available. Electrons in the Fermi band cannot find any lower energy states to become localized by as they travel through the sample. Under these conditions, a conducting electron at the cathode has, on average, the same energy as a conducting electron at the anode. Current can therefore travel without dissipating any energy, making the resistance effectively 0.

Increasing the magnetic field through the sample creates additional bound states with energy between the low and high bands. The high energy bound states occupy these while on the whole leaving the extended states the same. Eventually, the magnetic field is increased to the point where high-energy states are exhausted, and the Fermi band begins to donate electrons to become new low-energy localized states. Increasing the magnetic field beyond the point where Fermi electrons are exhausted causes the resistance to shoot up violently, since there are no more electrons to conduct in the sample. When the low-energy band of electrons is used up, another level of high-extended-low energy states opens up and the sample returns to a zero-resistance state once more.

0.2 PT-Quantum Mechanics

The Hall-Effect is a well-understood yet still relevant phenomenon that has distinct features that suit a non-hermitian approach. The first is that it is well-understood, and as such the predictions that a new theory will make are readily testable and verifiable. Second, certain non-hermitian theories predict phase transitions for systems

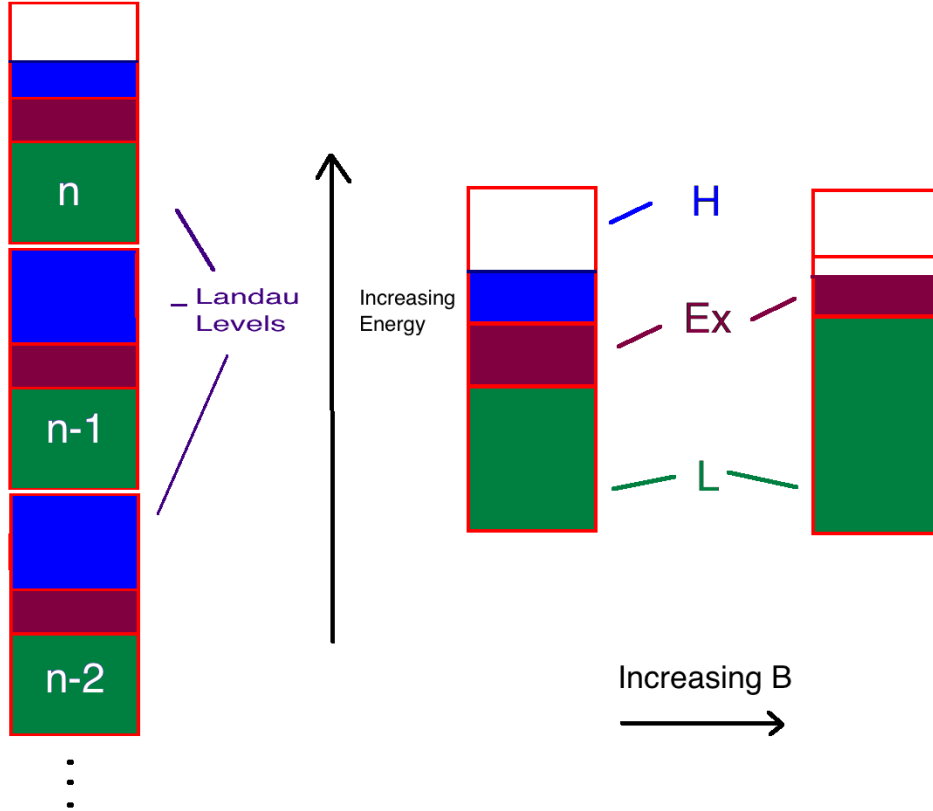


Figure 6: The band structure for electrons confined to a two dimensional layer with an applied magnetic field, including charged impurities. The blue band represents high energy states (H), green are low energy states (L), and red, extended states (Ex). Whitespace in the band structure indicates available space in a given band. As the perpendicular B field increases, the high energy band within the highest Landau level empties into low energy states.

with non-hermitian terms. Given that the hallmark (pun intended) of the effect is a series of phase transitions within the system makes the quantum Hall effect a great candidate system. Third, there are explanations utilizing Parity and Time-reversal quantum mechanics (PT-quantum mechanics or PTQM) to shed light on high temperature superconductivity and quantum magnetism[14]. As such, it ought to predict simpler phenomena, and may perhaps shed new light into the behavior of a seemingly well-understood problem (a point that Von Klitzing can surely relate to).

PT-quantum mechanics is a formulation of quantum mechanics that allows for certain non-hermitian operators to pertain to physical observables. This works by working over non-orthonormal bases (non-normalized bases). Holding operators to be self-adjoint enforces different rules according to the specific definition of the inner product. Over an orthonormal basis for ϕ of some vectors,

$$\langle n|m \rangle = \delta_{n,m},$$

making the wave function ϕ ,

$$|\phi\rangle = \sum_{i=1}^N a_n |n\rangle. \quad (4)$$

For a self-adjoint operator A ,

$$\langle A\phi|\psi\rangle = \langle\phi|A\psi\rangle, \quad (5)$$

with ϕ, ψ orthonormal implies that $A = (A^*)^T$. Choosing other bases for ϕ and requiring an operator A be self-adjoint implies a different set of criteria on A [15], and these do not restrict physical operators to be strictly equal to their conjugate transpose. PTQM has developed the theory into a self-consistent and reliable theory that has already predicted various unique phenomena. One of the major fields in which PTQM has made consistent predictions is within in the field of quantum optics, where Hamiltonian terms such as complex potentials pop up often. For such systems, PTQM has predicted numerous experimental results to an excellent degree of accuracy.

Chapter 1

Modeling a fermion in a magnetic field

1.1 Classical Motion

To understand how charged particles behave within a non-hermitian framework, we need a model to work on and to build a base case with which to compare the eventual results. We will begin by creating a classical picture of a particle traveling in the $x - y$ plane in a magnetic field. This classical model also yields an intuition about how our particles should behave under normal conditions.

According to the Lorentz force law, a classical particle experiences a force $\vec{F}$,

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}). \quad (1.1)$$

$$\Rightarrow \vec{F} = e[(\dot{y}B_z - \dot{z}B_y)\hat{x} + (\dot{z}B_x - \dot{x}B_z)\hat{y} + (\dot{x}B_y - \dot{y}B_x)\hat{z}] + \vec{E}.$$

Since $\vec{B} = \nabla \times \vec{A}$, where $\vec{A}$ is the magnetic vector potential, we find the x -component of the force as,

$$F_x = e \left[\dot{y} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \dot{z} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] + E_x = m\ddot{x}. \quad (1.2)$$

To simplify the force expression, assume a perpendicular magnetic field $B_z\hat{z}$, and that our electric field points in the $\hat{y}$ direction. According to Newton's second law,

$$F = m\vec{a} \Rightarrow m\ddot{x} = -eB_z\dot{y}, m\ddot{y} = eB_z\dot{x} + eE_y. \quad (1.3)$$

Solving Eq. 1.3 where $E \rightarrow 0$ yields the parametrization of a circle,

$$x[t] = A \cos[eBt/m + \delta], y[t] = A \sin[eBt/m + \delta]. \quad (1.4)$$

This is what we might expect for a particle moving in the presence of a strong magnetic field: it orbits at some radius A . Solving this equation for finite electric fields produces non-localized circles— orbits combined with linear motion, as shown in Figs. 1.1., 1.2. and 1.3.

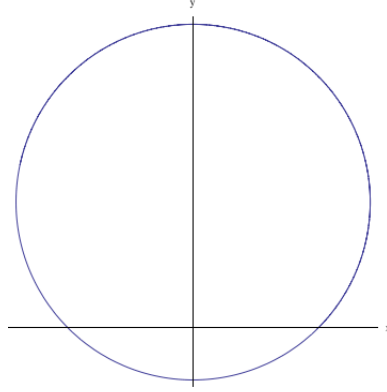


Figure 1.1: Classical motion for an electron in a strong magnetic field. The particle orbits at a fixed radius, and is localized.

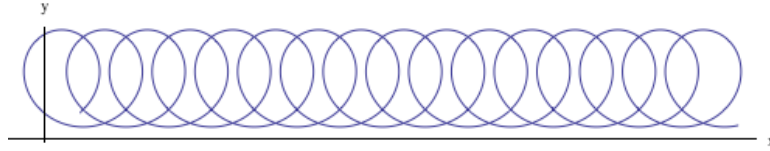


Figure 1.2: Classical motion for a particle in an electromagnetic field where E and B are perpendicular to each other. Notice that the net displacement of the particle is parallel with the electric field.

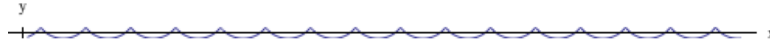


Figure 1.3: Classical motion for an electron in an electromagnetic field where E and B are perpendicular to each other, and E is strong compared with B .

We see that electrons can occupy a localized “orbit” and also travel relatively unhindered by the magnetic field, all depending on the relative field strengths.

The Lagrangian

The Lagrangian provides a bridge between classical and quantum regimes. The Lagrangian, L , is defined as

$$L = T - V$$

where T represents the kinetic energy and V represents some potential. For a particle in a magnetic field, the Lagrangian is

$$L = 1/2mv^2 - e(\phi + \mathbf{v} \cdot \mathbf{A}), \quad (1.5)$$

In the Quantum Hall regime, the electric field term ϕ is very small compared with A , and we can remove it, leaving the effective Lagrangian

$$L = 1/2mv^2 - e\mathbf{v} \cdot \mathbf{A}. \quad (1.6)$$

Before we continue, let's make sure that this system will reproduce the behavior that we found with the Lorentz force. The Euler-Lagrange equations are conditions that minimize the path that the particle may take given some conditions, yielding the equations of motion for such paths. The Euler-Lagrange equations say

$$\frac{\partial L}{\partial \alpha} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}}. \quad (1.7)$$

For the x-component we have

$$\begin{aligned} m\ddot{x} - e \frac{\partial A_x}{\partial t} &= -e \left(\dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + \dot{z} \frac{\partial A_x}{\partial z} \right), \\ \Rightarrow m\ddot{x} &= e \left(\frac{dA_x}{dt} - \dot{x} \frac{\partial A_x}{\partial x} - \dot{y} \frac{\partial A_x}{\partial y} - \dot{z} \frac{\partial A_x}{\partial z} \right). \end{aligned}$$

By the chain rule

$$\frac{dA_x}{dt} = \frac{d}{dt} A_x = \left(\frac{\partial}{\partial x} \frac{dx}{dt} + \frac{\partial}{\partial y} \frac{dy}{dt} + \frac{\partial}{\partial z} \frac{dz}{dt} \right) A_x. \quad (1.8)$$

This leaves,

$$m\ddot{x} = e \left[\dot{y} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \dot{z} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] \quad (1.9)$$

as in Eq. 1.2, confirming that our Lagrangian describes motion for the x -component according to the Lorentz force law and vice versa.

The Hamiltonian

We can use the Lagrangian to develop the Hamiltonian, a more useful tool in quantum mechanics. It is defined from the Lagrangian as follows:

$$H = \sum p_i \dot{q}_i - L. \quad (1.10)$$

With $L \approx \frac{1}{2}m\dot{x}^2 - e(\vec{v} \cdot \vec{A})$, Eq. 1.7 yields the velocity in terms of mechanical momentum,

$$\frac{\partial L}{\partial \dot{x}^\mu} = p^\mu = m\dot{x}^\mu - eA^\mu \rightarrow \dot{x}^\mu = \left[\frac{p^\mu + eA^\mu}{m} \right]. \quad (1.11)$$

Finally, the Hamiltonian can be written,

$$H = \frac{1}{2m}(p + eA)^2. \quad (1.12)$$

To study particles confined to a plane with a magnetic field normal to it, the vector potential should be vastly simplified. In the next step, choose the Landau gauge

$$A \rightarrow -Bx\hat{y}$$

called the Landau gauge. It confines the magnetic field to the $\hat{z}$ direction. The Hamiltonian now becomes,

$$H = \frac{1}{2m}(p_x\hat{x} + p_y\hat{y} - eBx\hat{y})^2 = \frac{1}{2m}(p_x^2 + p_y^2 - 2p_yeBx + e^2B^2x^2),$$

$$H = \frac{1}{2m}[p_x^2 + (p_y - eBx)^2]. \quad (1.13)$$

1.2 Quantum Mechanics

Up until now, the treatment has been entirely classical. The form of Eq. 1.13 allows for an easy transition between classical and quantum regimes. While in the classical case, the momenta were themselves parameters, in quantum mechanics the momenta, and the Hamiltonian itself is treated as an operator. The momentum parameter becomes the operator,

$$P = i\hbar\nabla. \quad (1.14)$$

For the momentum in a single cartesian direction, the momentum operator is,

$$p_x = \frac{\hbar}{i} \frac{\partial}{\partial x}. \quad (1.15)$$

According to the Schrödinger equation,

$$H\Psi = E\Psi. \quad (1.16)$$

The eigenvalues of the operator H acting on a wavefunction Ψ are some set of definite energies E . The energy spectrum is the set of energies that a particle with wavefunction Ψ can be measured to have. From Eq. 1.13, we can see that the form of our governing equation is simple in y , but not x . In this case, it makes sense to suggest the ansatz for the wave functions. Assume that the particle wave functions are plane waves along our y -axis, $\Psi_k(x, y) \rightarrow \phi_k(x)e^{ik_y y}$. One of the benefits of this substitution is clear when we find the eigenvalues of p_y ,

$$p_y[\Psi] = p_k\Psi,$$

$$\frac{\hbar}{i} \frac{\partial}{\partial y} [e^{ik_y y} \phi(x)] = p_k e^{ik_y y} \phi(x)$$

$$\Rightarrow p_k = \hbar k_y. \quad (1.17)$$

We can substitute this into our Hamiltonian to make it one-dimensional for some constant parameter k_y , a great bonus. From Eq. 1.13,

$$\frac{1}{2m}p_x^2\phi(x) + \frac{1}{2}m(\hbar k_y - eBx)^2\phi(x) = E\phi(x). \quad (1.18)$$

Now, define the magnetic length

$$l = \sqrt{\frac{\hbar}{eB}}$$

, and the cyclotron frequency

$$\omega_c = eB/m$$

. Plugging these in yields

$$\frac{1}{2m}p_x^2\phi(x) + \frac{1}{2}m\omega_c^2(x - kl^2)^2\phi(x) = E\phi(x). \quad (1.19)$$

Converting the momentum to its operator form in Eq. 1.15. reduces the Hamiltonian to differential equation in x, k ,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}m\omega_c^2(x - kl^2)^2\phi(x) = E_n\phi(x). \quad (1.20)$$

To create a neat graph of the wave functions, it helps to non-dimensionalize Eq. 1.20. To solve this equation analytically, define a useful dimension x' as,

$$lx' = x - kl^2. \quad (1.21)$$

Note that x' is a non-dimensional variable. Plugging our relevant parameters into the equation yields,

$$\begin{aligned} -\frac{\hbar^2}{2ml^2}\frac{\partial^2\phi}{\partial x'^2} + \frac{1}{2}m\omega_c^2(lx')^2\phi(x) &= E_n\phi(x), \\ -\frac{\hbar\omega_c}{2}\frac{\partial^2\phi}{\partial x'^2} + \frac{\hbar\omega_c}{2}(x')^2\phi(x) &= E_n\phi(x'). \end{aligned}$$

Making the nifty non-dimensionalization $E = E_0\epsilon$, and choosing $E_0 = \hbar\omega_c$ simplifies the expression to,

$$-\frac{1}{2}\frac{\partial^2\phi}{\partial x'^2} + \frac{x'^2}{2}\phi(x') = \epsilon\phi(x'). \quad (1.22)$$

Using this method yields a class of n solutions with the accompanying condition on the energy of the system:

$$\phi_n(x') = AH_n(x')e^{-x'^2/2}, \epsilon_n = (n + 1/2), \quad (1.23)$$

where $H_n(x')$ is the n^{th} Hermite polynomial, and the normalization constant A is found to be (See Appendix A),

$$A = \frac{1}{\sqrt{2^n n! \pi^{1/2}}}. \quad (1.24)$$

Replacing x for x' from Eq. 1.21. yields

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \pi^{1/2}}} H_n(x - kl^2) e^{-(x - kl^2)^2/2l^2} \quad (1.25)$$

$$\phi_n(x, y) = \frac{1}{\sqrt{2^n n! \pi^{1/2}}} e^{iky - (x - kl^2)^2/2l^2} H_n(x - kl^2). \quad (1.26)$$

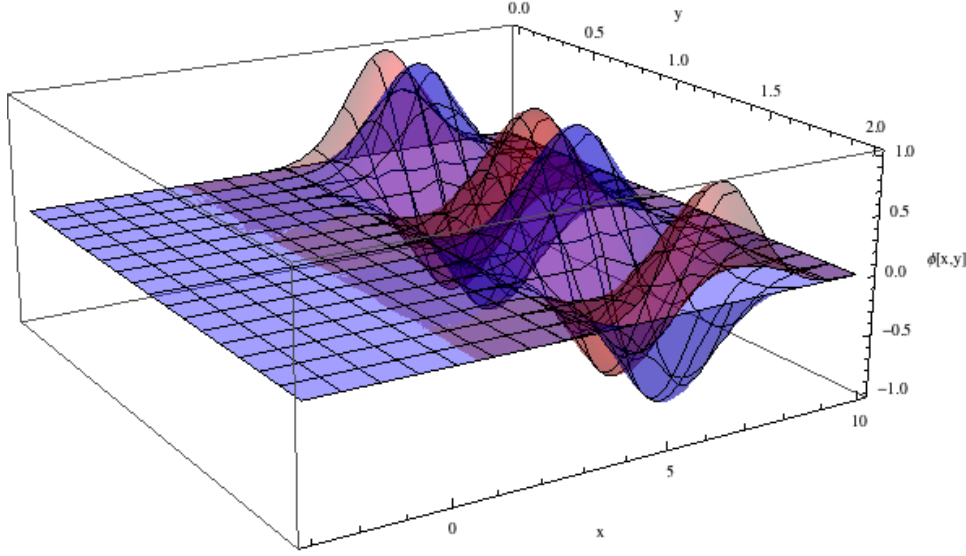


Figure 1.4: The real (red) and imaginary (blue) parts of the $n = 1$, $k=2\pi/L$ wave function $\Psi_1[x, y]$. Here l is set to 1 for simplicity

If the y -axis has periodic boundaries, then we know that $k_P \rightarrow 2\pi P/L$, where P is an integer. n is called the Landau level– a highly degenerate grouping of states that have the same energy. A plot of the wave function from Eq. 1.26 shown in Fig 1.2.

Before continuing from this picture, take note of the dimensionalized energy eigenvalues,

$$E_n = E_0 \epsilon_n = \hbar \omega_c (n + 1/2). \quad (1.27)$$

The energy levels are quantized, defining the allowed energy that a particle can be measured to have. Further, these energy levels depend on the cyclotron frequency ω_c , which is proportional to the magnetic field. As we increase the magnetic field, we expect the energy of each level to increase. Changing the magnetic field also changes the magnetic length and therefore the wave functions too. Note that the increase of E_0 is a B -dependent parameter, while l is also B -dependent. To get Hall quantization, we need a new parameter that describes the system in terms of its Landau levels.

The Filling Factor: (ν)

Assuming that particles are evenly distributed within the sample, how many particles exist in a given Landau level? As is clear from the form of Eq. 1.26., the centers of bounded states are shifted in $\hat{x}$. The wave function for given k, l values is centered at a value of x such that

$$x - kl^2 = 0. \quad (1.28)$$

Imposing periodic boundary conditions on the $\hat{y}$ direction of the wave function enforces a condition for state spacing in the x -direction. For such a condition we find

$$e^{ik_y(y+L_y)} = e^{ik_y y} \quad (1.29)$$

implying

$$k_y = \frac{2\pi P}{L_y}. \quad (1.30)$$

Now consider some length L_x in $\hat{x}$. Within this length there are N states, and Eq. 1.28 gives

$$L_x = \frac{2\pi N l^2}{L_y} \Rightarrow N = \frac{L_x L_y}{2\pi l^2}. \quad (1.31)$$

The number of states per unit area (degeneracy per unit area) comes readily from this equation,

$$D = \frac{N}{L_x L_y} = \frac{1}{2\pi l^2}. \quad (1.32)$$

We can now define the filling factor ν , the number of particles in a Landau level, in terms of the particle density ρ :

$$\nu = \frac{\rho}{D} = 2\pi l^2 \rho. \quad (1.33)$$

For the n^{th} Landau level, the current density $J(r)$ is

$$J(r) = -ne\rho(r)v(r), \quad (1.34)$$

eventually leading to the longitudinal current (See Appendix A.0.2).

$$\hat{I}_y = -ne \int \frac{dp_y}{2\pi\hbar} \left\langle \frac{\partial H}{\partial p_y} \right\rangle \quad (1.35)$$

$$\hat{I}_y = -\frac{ne}{h} \int \frac{\partial E}{\partial x} \quad (1.36)$$

$$\hat{I}_y = \frac{ne^2}{h} V_H \quad (1.37)$$

Finally retrieving Eq. 3., the Hall resistance for n filled Landau levels is,

$$R_H = \frac{V_H}{I_y} = \frac{h}{ne^2}. \quad (1.38)$$

1.3 Square Lattice

We can extend the analytic system to a discrete lattice. For a square lattice with spacing a , the tight-binding model yields the Bloch energy function[6],

$$W(\vec{k}) = 2E_0(\cos(k_x a) + \cos(k_y a)). \quad (1.39)$$

We make the famous Peirels substitution,

$$\hbar k = \vec{p} - e\vec{A}, \quad (1.40)$$

$$\Rightarrow k_y = \frac{p_y - eA_y}{\hbar}, k_x = \frac{p_x}{\hbar}. \quad (1.41)$$

Plugging this in to the Bloch energy function gives

$$W(\vec{k}) = E_0((e^{\frac{ia p_x}{\hbar}} + e^{-\frac{ia p_x}{\hbar}}) + (e^{\frac{ia p_y}{\hbar} - \frac{ie B x a}{\hbar}} + e^{-\frac{ia p_y}{\hbar} + \frac{ie B x a}{\hbar}})). \quad (1.42)$$

Eq. 1.42 contains the canonical translation operators,

$$T(+a) = e^{\frac{ia}{\hbar} p},$$

$$T(-a) = e^{-\frac{ia}{\hbar} p}.$$

These operators, when acting on a wave function, shift the wave function by a lattice position like so,

$$T(+a)\Psi(\beta) = \Psi(\beta + a). \quad (1.43)$$

The equation thus becomes,

$$W(\vec{k}) = E_0 \left(T(+a_x) + T(-a_x) + e^{-\frac{ie B a x}{\hbar}} T(+a_y) + e^{\frac{ie B a x}{\hbar}} T(-a_y) \right). \quad (1.44)$$

Whose eigenvalues are then given by,

$$E_0(\Psi(x + a, y) + \Psi(x - a, y) + e^{-\frac{ie B a x}{\hbar}} \Psi(x, y + a) + e^{\frac{ie B a x}{\hbar}} \Psi(x, y - a)) = E \Psi(x, y). \quad (1.45)$$

The form of Eq. 1.45 allows us to capture both B and l dependence at the same time. To this end, define a new parameter α ,

$$\alpha = \frac{a^2}{2\pi l^2} = \frac{a^2 e B}{2\pi \hbar}. \quad (1.46)$$

In this case, α will be the stand-in for magnetic field strength as it defines the strength of the magnetic field relative to the size of the lattice. We can see that as it incorporates Eq. 1.32 and the lattice size, it is also the number of wave functions in each lattice square. As we saw in Eq. 1.26, the wave functions and energy levels should depend on the magnetic field and the size of our system to the magnetic field. The parameter α combines the two into a single parameter.

On a lattice, $x = ma$ and $y = na$ for some integers $\{m, n\} > 0$. To model plane-waves in the y -direction, we have the square lattice equivalent to the ansatz in Eq. 1.17,

$$\Psi(ma, na) = \phi(m) e^{i\eta n}.$$

When plugged in,

$$e^{i\eta n} (\phi(m+1) + \phi(m-1) + (e^{-2im\pi\alpha + i\eta} + e^{2im\pi\alpha - i\eta})\phi(m)) = \epsilon e^{i\eta n} \phi(m),$$

Finally yielding “Harper’s equation”,

$$\phi(m+1) + \phi(m-1) + 2\cos[2m\pi\alpha - \eta]\phi(m) = \epsilon\phi(m). \quad (1.47)$$

To calculate ϵ for lattices of interesting size, we use a computer. We are interested in how the set of ϵ changes as we vary our parameter α . The very intricate structure of ϵ as α is varied is shown in Fig. 1.5.

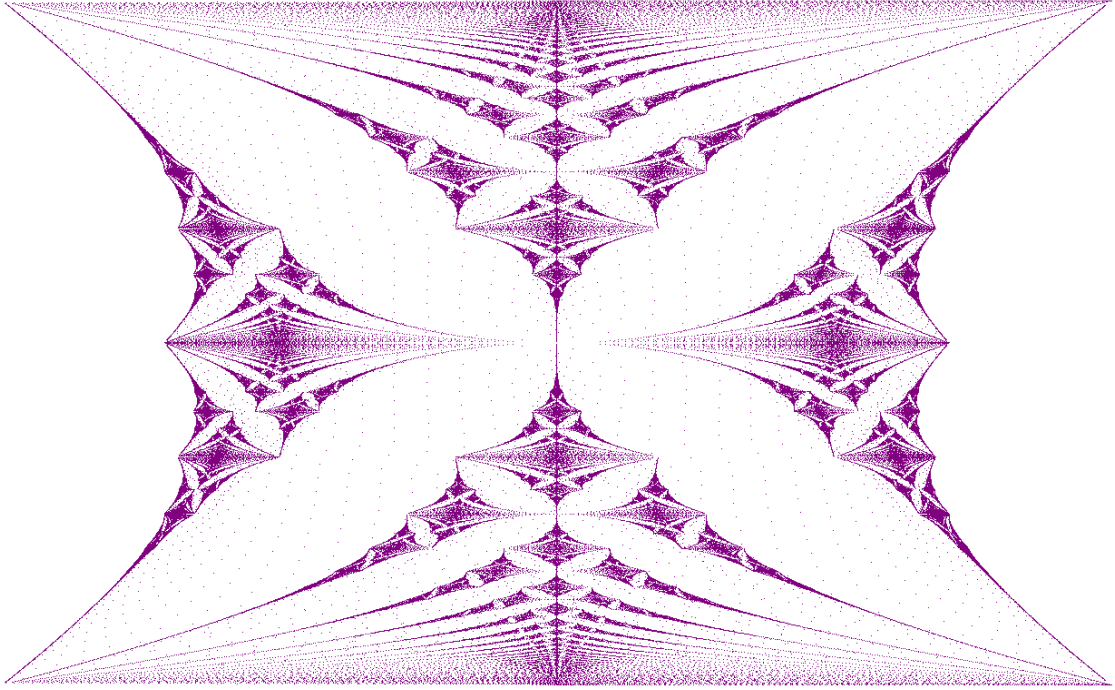


Figure 1.5: The eigenvalue spectrum for Harper’s equation with $\alpha \in [0, 1]$, called the Hofstadter butterfly[6]. The vertical axis varies α , while the horizontal axis indicates ϵ (which in this case shows the interval $[-4, 4]$).

Chapter 2

The Non-Hermitian Hamiltonian

2.1 Theory

An operator A is hermitian if it is equal to its own conjugate transpose,

$$A = (A^*)^T = A^\dagger. \quad (2.1)$$

The inner product of two hermitian states Ψ and Φ is,

$$\langle \Psi | \Phi \rangle = \Psi^\dagger \Phi = (\Psi^*)^T \Phi. \quad (2.2)$$

The eigenvalues of are the possible results of measurement[1],and hermitian operators are guaranteed to have real eigenvalues. Furthermore, hermitian operators give rise to unitary time evolution,

$$|\Psi(t)\rangle = e^{-iHt/\hbar} |\Psi_0\rangle. \quad (2.3)$$

Unitarity guarantees that the norm of a wave function be invariant under time evolution (also known as probability conservation).

Conventional quantum mechanics asserts that observables pertain to hermitian operators [1]. Despite the enormous successes of quantum mechanics, hermitian quantum mechanics fails to describe physical systems with real, positive energy spectra[15]. For example, the non-hermitian Hamiltonian,

$$H = p^2 + m^2 x^2 - (ix)^N \quad (2.4)$$

has all-real energy eigenvalues when N is a non-negative integer. To account for physical yet non-hermitian Hamiltonians theories of non-hermitian quantum mechanics (NHQM) relax the condition of hermiticity on operators representing observables.

2.1.1 PT-Quantum Mechanics

Parity and Time-Reversal quantum mechanics (PTQM) is a contemporary, self-consistent non-hermitian quantum theory that maintains conventional quantum theory within

a less-restrictive framework. PTQM systems exhibit strange and unexpected properties, and it has predicted several striking physical phenomena, most notably a novel phase transition in non-hermitian systems.

Instead of imposing hermiticity on operators, PTQM reverts to the less restrictive condition that operators be self-adjoint with respect to the inner product. Since a self-adjoint inner product is hermitian when evaluated over an orthonormal basis, evaluating over an orthogonal basis allows certain non-hermitian operators to pertain to observables. We can represent the action of the Parity operator with a matrix S : $P\Psi = S\Psi$, and the action of the Time-reversal operator with complex conjugation, $T\Psi = \Psi^*$. As such, PTQM can use the Parity and Time operators themselves to enforce the condition of a self-adjoint inner product space,

$$(\Psi|\Phi) \equiv (PT\Psi)^T\Phi = \Psi^\dagger S\Phi = \Psi^\dagger P\Phi. \quad (2.5)$$

Quantum theories are typically formulated over orthonormal bases where each unit vector is perpendicular to the other and has length 1. For two such basis vectors f_i, f_j , their inner product is,

$$\langle f_i|f_j \rangle = \delta_{ij}. \quad (2.6)$$

In PTQM, the same calculation becomes

$$(f_i|f_j) = k[x]\delta_{ij}. \quad (2.7)$$

Notice that the basis is orthogonal, but not normalized. This simple reconfiguration allows for the consideration of non-hermitian operators that adhere to the following criteria[16],

1. Invariance Under PT operators (ex: $[H, PT] = 0$).
2. Simultaneous eigenfunctions of H and PT .
3. Self-adjointness WRT the new inner product $(f_i|f_j)$.

A Hamiltonian is said to be PT-symmetric if the eigenfunctions of H , and PT are equal. If they aren't then the system is said to have broken PT-symmetry.

2.1.2 PT-Phase Transition

The theoretical framework of PT-quantum mechanics predicts that certain systems can break or exhibit PT-symmetry depending on the values of non-hermitian parameters within the system[3]. For instance, a Hamiltonian with non-hermitian term proportional to g might produce a transition from all-real eigenvalues to a complex eigenvalue spectrum for some value of g . This fundamental change in the character of the eigenvalue spectrum is called a PT-phase transition. At these transitions, at least one eigenvalue meets another eigenvalue on the real plane and the pair are ejected into the complex plane[17]. The physical significance of a PT-phase transition is still not well-known, and thus understanding the mechanics of such behavior is of great interest.

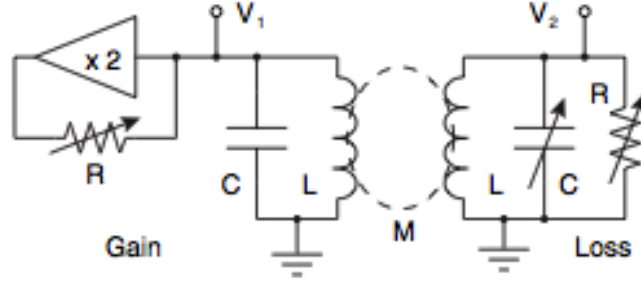


Figure 2.1: Coupled LRC circuit exhibiting PT-phase transition according to Eq. 2.10. (Bittner et al., Phys Rev. Lett 108, 2 024101 (2012))

Consider the non-hermitian Hamiltonian matrix describing a system made of both an energy source and an energy well coupled together by constant g (see [18] for an in-depth description). Solving the Schrödinger equation for this system yields the equations,

$$ae^{i\theta}\psi_1 + g\psi_2 = \epsilon\psi_1 \quad (2.8)$$

$$g\psi_1 + ae^{-i\theta}\psi_2 = \epsilon\psi_2 \quad (2.9)$$

The matrix representation of these equations yields the Hamiltonian

$$H = \begin{pmatrix} ae^{i\theta} & g \\ g & ae^{-i\theta} \end{pmatrix}. \quad (2.10)$$

The reality of the spectrum of H depends on the coupling constant, g . For $g^2 < a^2 \sin^2(\theta)$, the eigenvalues of H are complex, and the system is said to be in a broken PT phase. However, for $g^2 > a^2 \sin^2(\theta)$, the eigenvalues of H are all real, and H is said to be in an unbroken PT phase. At $g^2 = a^2 \sin^2(\theta)$ the system transitions between PT-symmetric and broken PT-symmetry regions, called a PT-phase transition.

Characteristics of PT-phase transitions

Eq. 2.10. is a theoretical description of at least three separate physical systems. Schindler et al. implemented an LRC-equivalent coupled pair of LRC circuits[19]. Bittner et al. reconstructed an equivalent optical system of the Hamiltonian with coupled lossy and pumped waveguides and a microwave cavity[17]. After these experiments, Bender constructed a simple coupled pendulum system offering an accessible example of a PT-phase transition[18]. Each of these experiments has reproduced the theorized PT-phase transition to an excellent degree of accuracy.

These simple systems commensurate with Eq. 2.10. display several commonalities. Each system describes a coupled system where energy is exchanged between oscillators. The coupling term determines the stability of the solution. For certain regimes of the coupling term, the system is forced into or out of energetic equilibrium (breaking PT-symmetry). When the system breaks PT symmetry, the system runs off in a pair of negative and positive gain states, similar to a diverging wave function.

Chapter 3

Momentum Perturbation

PT-quantum mechanics allows us to consider an entirely new category of Hamiltonians that differ significantly with hermitian theories. Adding complex perturbations to the Hamiltonian will take our system into the regions where hermitian theories do not reach, while keeping a clear base case. Perturbing the hermitian Hamiltonian H should yield a non-hermitian perturbation Hamiltonian H_i :

$$H \rightarrow H_0 + \beta H_i. \quad (3.1)$$

For reasons that will become clear, I propose to make the following momentum perturbation motivated by the results of Nelson and Hatano[7],

$$P \rightarrow P + i\beta. \quad (3.2)$$

Adding a complex constant to the momentum has a few different physical interpretations. The complex perturbation could come about through moving through some small complex vector field[20], for instance. Although thoroughly non-hermitian, this change produces real energy spectra for a range of parameters in similar lattice Hamiltonians.

The edit is simple, it produces a PT-phase transition in some system, and as we will discover, the Nelson-Hatano system is very similar in form to the resulting non-hermitian Harper's equation (Eq. 3.22). What follows is a summary of the relevant results of Nelson and Hatano's studies of a particle on a one-dimensional lattice with a random potential[7][20][21].

A Non-Hermitian Hamiltonian with a Random Potential

The Hamiltonian for a particle in a single-dimension with both a random potential and the perturbation from Eq 3.2. is

$$H = \frac{(P_x + ig)^2}{2m} + V(x). \quad (3.3)$$

While Nelson and Hatano provide a lattice version of this Hamiltonian,

$$H_{x,x'} = -\frac{t}{2} (e^{\bar{g}} \delta_{x,x'+1} + e^{-\bar{g}} \delta_{x,x'-1}) + V_x \delta_{x,x'}, \quad (3.4)$$

this lattice Hamiltonian is not derived in an equivalent way to the Harper Hamiltonian. To stay consistent, we are motivated to reformulate this Hamiltonian according to the Bloch energy theorem method used by Hofstadter. As before, the Bloch energy theorem forces the energy function $W(\vec{k}_x)$ to have the form,

$$W(\vec{k}_x) = 2E_0 \cos[k_x a] + E_0 V(x). \quad (3.5)$$

Making the same identifications in Eqs. 1.40. and 1.41. yields the familiar form for the energy,

$$W(\vec{k}_x) = E_0 \left(e^{i \frac{P_x a}{\hbar}} + e^{-i \frac{P_x a}{\hbar}} \right) + V(x). \quad (3.6)$$

Finding the energy eigenvalues and non-dimensionalizing yields,

$$\left(e^{i \frac{P_x a}{\hbar}} \Psi(x) + e^{-i \frac{P_x a}{\hbar}} \Psi(x) \right) + V(x) \Psi(x) = \epsilon \Psi(x). \quad (3.7)$$

Making the non-hermitian transition $P_x \rightarrow P_x + ig$ yields,

$$\left(e^{-\frac{ga}{\hbar}} e^{i \frac{P_x a}{\hbar}} \Psi(x) + e^{\frac{ga}{\hbar}} e^{-i \frac{P_x a}{\hbar}} \Psi(x) \right) + V(x) \Psi(x) = \epsilon \Psi(x). \quad (3.8)$$

There are two important things to note: first, the leading exponential terms containing g can be simplified by absorbing the common factor into a new constant $\bar{g}$ leaving the non-dimensionalized exponent,

$$e^{\frac{ga}{\hbar}} = e^{\bar{g}}, \quad (3.9)$$

As we will see later, producing the perturbed Hofstadter butterfly requires non-dimensionalizing g to retrieve a non-trivial factor. Since a and $\hbar$ and g do not relate to the system in any overarching way, we can get by without non-dimensionalizing since g is just a constant. Further, this eigensystem (Eq. 3.8) includes the same canonical translation operators as introduced in Eq. 1.43. Replacing these and noting $x = ma$, we get,

$$e^{-\bar{g}} \phi(m+1) + e^{\bar{g}} \phi(m-1) + V(m) \phi(m) = \epsilon \phi(m). \quad (3.10)$$

Now examine the form of Eq. 3.10. in relation to Eq. 3.4. For $t = 2$, $V(m) = V_{x/a}$ in the range $[-1, 1]$ (the exact system parameters from [7]), these treatments are simply negatives of each other and should therefore yield a reversed eigenvalue spectrum.

Note that the Nelson-Hatano system can be broken into the form of Eq. 3.1 by defining,

$$H_0 = \Psi(m+1) + \Psi(m-1) + V(m) \Psi(m), \quad (3.11)$$

and,

$$\beta H_i = ([e^{-\bar{g}} - 1] \Psi(m+1) + [e^{\bar{g}} - 1] \Psi(m-1)). \quad (3.12)$$

$$H_5 = \begin{pmatrix} \in[-1,1] & e^{-g} & 0. & 0. & e^g \\ e^g & \in[-1,1] & e^{-g} & 0. & 0. \\ 0. & e^g & \in[-1,1] & e^{-g} & 0. \\ 0. & 0. & e^g & \in[-1,1] & e^{-g} \\ e^{-g} & 0. & 0. & e^g & \in[-1,1] \end{pmatrix}$$

Figure 3.1: The “Hamiltonian” matrix for the Bloch energy formulation of the Nelson-Hatano system on a 5-site, continuous lattice.

$$H_0 = \begin{pmatrix} \in[-1,1] & 1 & 0. & 0. & 1 \\ 1 & \in[-1,1] & 1 & 0. & 0. \\ 0. & 1 & \in[-1,1] & 1 & 0. \\ 0. & 0. & 1 & \in[-1,1] & 1 \\ 1 & 0. & 0. & 1 & \in[-1,1] \end{pmatrix}$$

Figure 3.2: The unperturbed “Hamiltonian” matrix for the Bloch energy formulation of the Nelson-Hatano system on a 5-site, continuous lattice.

$$\beta H_i = \begin{pmatrix} 0. & e^{-g} - 1 & 0. & 0. & e^g - 1 \\ e^g - 1 & 0. & e^{-g} - 1 & 0. & 0. \\ 0. & e^g - 1 & 0. & e^{-g} - 1 & 0. \\ 0. & 0. & e^g - 1 & 0. & e^{-g} - 1 \\ e^{-g} - 1 & 0. & 0. & e^g - 1 & 0. \end{pmatrix}$$

Figure 3.3: The perturbation “Hamiltonian” matrix for the Bloch energy formulation of the Nelson-Hatano system on a 5-site, continuous lattice.

The Non-Hermitian Quantum Hall Effect

Now that we have a reference system for a non-hermitian momentum perturbation, we can create a non-hermitian quantum Hall system. We begin with the basic Hamiltonian in Eq. 1.13. Making the transition $P \rightarrow P + i\beta$ (meaning $P_x \rightarrow P_x + ig$ and $P_y \rightarrow P_y + iq$) gives

$$H + H_j = \frac{1}{2m} ((P_x + ig)^2 + ((P_y + iq) - eBx)^2), \quad (3.13)$$

$$H + H_j = \frac{1}{2m} (P_x^2 + (P_y - eBx)^2) + \frac{1}{2m} (-(g^2 + q^2) + 2i(gP_x + q(P_y - eBx))).$$

For small values of g and q , $g^2 \approx q^2 \approx 0$. Introducing the operator forms for our momenta,

$$H + H_j = \frac{1}{2m} \left(-\hbar^2 \frac{\partial^2}{\partial x^2} + (\hbar k - eBx)^2 \right) + \frac{1}{2m} \left(2i(g \frac{\hbar}{i} \frac{\partial}{\partial x} + q(\hbar k - eBx)) \right).$$

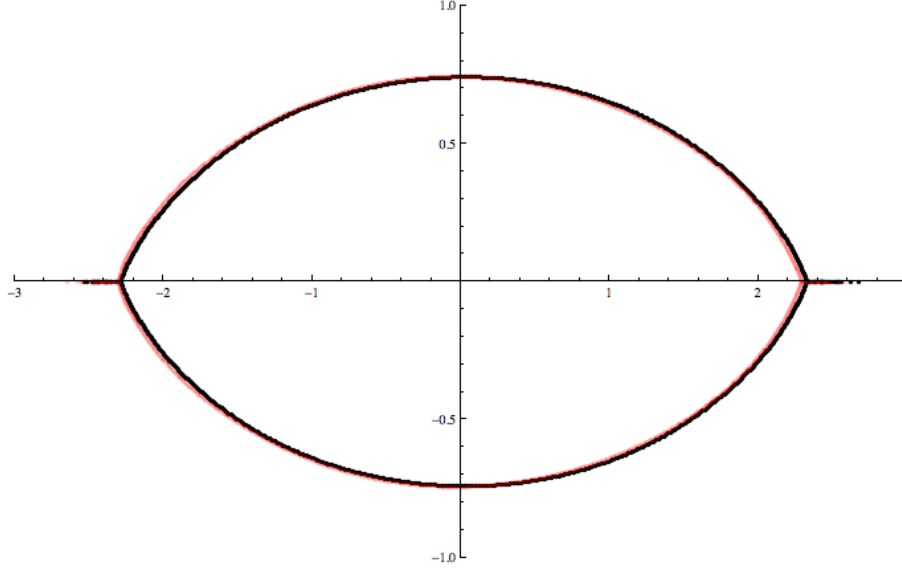


Figure 3.4: $\text{Im}[\epsilon]$ (vertical axis) vs. $\text{Re}[\epsilon]$ (horizontal axis) for $n = 1000$, $g = 0.4$ in the Nelson and Hatano system[7]. The red line indicates the eigenvalue spectrum for Eq. 3.10., while the black indicates the exact formulation from their original publication (Eq. 3.4.). The red line spectrum is blurred and opaque to show the correspondence between the two spectra.

As in the derivation of Eq. 1.19, introduce the following parameters in addition to non-dimensionalizing our non-hermitian perturbations:

$$\begin{aligned}\omega_c &= \frac{eB}{m} \\ l &= \sqrt{\frac{\hbar}{eB}} \\ lx' &= x - \hbar k / eB \\ g &= g' \sqrt{eB\hbar}, q = q' \sqrt{eB\hbar}\end{aligned}\tag{3.14}$$

This makes the perturbed Hamiltonian

$$H + \beta H_i = -\frac{\hbar\omega_c}{2} \frac{\partial^2}{\partial x'^2} + \frac{\hbar\omega_c}{2} x'^2 + \hbar\omega_c g' \frac{\partial}{\partial x'} + iq' \hbar\omega_c x'.\tag{3.15}$$

With $E = E_0\epsilon$ where $E_0 = \hbar\omega_c$ the eigenvalue equation dictates,

$$\begin{aligned}(H + \beta H_i)\phi(x') &= E\phi(x') \\ \rightarrow -\frac{1}{2} \frac{\partial^2 \phi'}{\partial x'^2} + \frac{1}{2} x'^2 \phi' + g' \frac{\partial \phi'}{\partial x'} + iq' x' \phi' &= \epsilon \phi'.\end{aligned}\tag{3.16}$$

This equation has solution

$$\phi(x') = e^{-x'^2/2 + (g' + iq')x'} \text{Hermite}\left[n - \frac{g'^2 + q'^2}{2}, x' + iq'\right]$$

$$\Rightarrow \phi(x') \approx e^{-x'^2/2+(g'+iq')x'} H_n[x' + iq'].$$

Replacing $x' = x/l - kl$ with the non-dimensionalized $X = x/l$ yields,

$$\phi(X) = e^{-\frac{(X-kl^2)^2}{2}+(g'+iq')(X-kl^2)} H_n[X - kl^2 + iq']. \quad (3.17)$$

To examine the behavior of the g and q perturbations, the g parameter needs to be held constant while q is varied and vice-versa. Setting the non-examined variable to 0 yields two equations:

$$\phi_g(X) = e^{-\frac{(X-kl^2)^2}{2}+g(X-kl^2)} H_n[X - kl^2], \quad (3.18)$$

$$\phi_q(X) = e^{-\frac{(X-kl^2)^2}{2}+iq(X-kl^2)} H_n[X - kl^2 + iq]. \quad (3.19)$$

The analytic eigenvectors for $\phi(x)$ are shown in Fig. 5.0.1.

What happens to the eigenvalues under the momentum perturbation in Eq. 3.2? Substituting $P_x \rightarrow P_x + ig$, $P_y = \hbar k_y \rightarrow \eta + iq$ the $\hat{y}$ wave function cancels out as before,

$$\begin{aligned} & E_0(e^{\frac{ag}{\hbar}}\Psi(x+a, y) + e^{-\frac{ag}{\hbar}}\Psi(x-a, y) \\ & + e^{-\frac{ieBxa}{\hbar}+\frac{aq}{\hbar}+in}\Psi(x, y+a) + e^{\frac{ieBxa}{\hbar}-\frac{aq}{\hbar}-in}\Psi(x, y-a)) \\ & = E\Psi(x, y) \end{aligned} \quad (3.20)$$

We want our equation in terms of non-dimensionalized, non-hermitian perturbations and α . To this end, note that,

$$\frac{ag}{\hbar} = \frac{a\bar{g}\sqrt{eB\hbar}}{\hbar} = \bar{g}\sqrt{2\pi\alpha}. \quad (3.21)$$

Applying the same logic to the perturbation q and canceling the overall $e^{in\eta}$ term, we find the perturbed Hamiltonian,

$$e^{\bar{g}\sqrt{2\pi\alpha}}\phi(m+1) + 2\cos[2m\pi\alpha - \eta - i\sqrt{2\pi\alpha}\bar{q}]\phi(m) + e^{-\bar{g}\sqrt{2\pi\alpha}}\phi(m-1) = \epsilon\phi(m). \quad (3.22)$$

$$H_3 = \begin{pmatrix} 2\cos(2\pi\alpha - \eta - i\sqrt{2\pi\alpha}\bar{q}) & e^{\sqrt{2\pi\alpha}\bar{g}} & e^{\sqrt{2\pi\alpha}\bar{g}} \\ e^{\sqrt{2\pi\alpha}\bar{g}} & 2\cos(4\pi\alpha - \eta - i\sqrt{2\pi\alpha}\bar{q}) & e^{\sqrt{2\pi\alpha}\bar{g}} \\ e^{\sqrt{2\pi\alpha}\bar{g}} & e^{\sqrt{2\pi\alpha}\bar{g}} & 2\cos(6\pi\alpha - \eta - i\sqrt{2\pi\alpha}\bar{q}) \end{pmatrix}$$

Figure 3.5: The 3x3 Hamiltonian describing the perturbed Harper's equation for a three site lattice in Eq. 3.22.

Note that Eq. 3.22. can be converted into the form of Eq. 3.1. for $\bar{q} = 0$ for H_0 and H_i defined,

$$H_0 = \phi(m-1) + \phi(m+1) + 2\cos[2m\pi\alpha - \eta]\phi(m), \quad (3.23)$$

and

$$\beta H_i = \left[e^{\bar{g}\sqrt{2\pi\alpha}} - 1 \right] \phi(m-1) + \left[e^{-\bar{g}\sqrt{2\pi\alpha}} - 1 \right] \phi(m+1). \quad (3.24)$$

Enforcing $\bar{q} = 0$ to maintain the general form of a non-hermitian perturbation hints at an inherent problem with the $\hat{y}$ perturbation. We will return to that instability for $\bar{q}$ in the results section.

A cursory examination of Eqs. 3.23. and 3.24. in relation to Eqs. 3.11. and 3.12. shows how very similar the two systems are. Both the perturbations and the ground states of each take the same basic form, with the only difference being the choice of potential (random versus deliberate). Because the Nelson-Hatano system exhibits such a characteristic and clean phase transition, this indicates that our system has a good chance of exhibiting non-hermitian behavior and possibly a phase transition. The hint of interesting non-hermitian is despite the general consensus that the quantum Hall Hamiltonian exhibits no unusual non-hermitian behavior, with at least one published source maintaining that the Harper system is robust against non-hermitian perturbation[22].

Chapter 4

Computation

4.1 Energy Eigenvalues

From the Harper's equation (Eq. 3.22.), we have the following set of equations,

$$\begin{aligned}\phi_2 + 2 \cos[2\pi\alpha - \eta]\phi_1 + \phi_n &= \epsilon\phi_1, \\ \phi_3 + 2 \cos[4\pi\alpha - \eta]\phi_2 + \phi_1 &= \epsilon\phi_2, \\ &\dots \\ \phi_1 + 2 \cos[2n\pi\alpha - \eta]\phi_n + \phi_{n-1} &= \epsilon\phi_n.\end{aligned}$$

For an n -dimensional lattice, the energy eigenvalues for non-dimensionalized H satisfy the following equation,

$$H\phi = \epsilon\phi, \quad (4.1)$$

where ϵ is simply some number. H can be written in matrix form,

$$H = \begin{pmatrix} 2 \cos[2\pi\alpha - \eta] & 1 & 0 & \dots & 1. \\ 1 & 2 \cos[4\pi\alpha - \eta] & 1 & \dots & \dots \\ 0. & \dots & \dots & \dots & \dots \\ \dots & \dots & 1 & 2 \cos[2(n-1)\pi\alpha - \eta] & 1 \\ 1. & \dots & \dots & 1 & 2 \cos[2n\pi\alpha - \eta] \end{pmatrix}. \quad (4.2)$$

as well as the ϕ vector,

$$|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \dots \\ \phi_n \end{pmatrix}. \quad (4.3)$$

To find the eigenvalues of H in Eq. 3.22 for various lattice sizes, we need to create the Hamiltonian matrix to find the locations of phase transitions for both $\bar{q}$ and $\bar{g}$. The general form of the perturbed energy system matrix is

$$H = \begin{pmatrix} 2 \cos[2\pi\alpha - \eta - \bar{q}\sqrt{2\pi\alpha}] & e^{-\bar{g}\sqrt{2\pi\alpha}} & \dots \\ e^{\bar{g}\sqrt{2\pi\alpha}} & 2 \cos[4\pi\alpha - \eta - \bar{q}\sqrt{2\pi\alpha}] & e^{-\bar{g}\sqrt{2\pi\alpha}} \dots \\ \dots & \dots & \dots \end{pmatrix}, \quad (4.4)$$

```

hofstadter[α_, q_, η_, n_, g_] := Module[{H, k, t},
  H = Table[0.0, {i, n}, {j, n}];
  H[[1, n]] = 1;
  H[[1, 2]] = 1;
  H[[1, 1]] = 2 * Cos[2 * Pi * α - I * Sqrt[2 * Pi * α] * q - η];
  For[k = 2, k ≤ n - 1, k = k + 1,
    H[[k, k + 1]] = 1;
    H[[k, k]] = 2 * Cos[2 * Pi * k * α + -I * Sqrt[2 * Pi * α] * q - η];
    H[[k, k - 1]] = 1;];
  H[[n, 1]] = 1;
  H[[n, n - 1]] = 1;
  H[[n, n]] = 2 * Cos[2 * Pi * n * α - I * Sqrt[2 * Pi * α] * q - η];
  Return[H];

newnnh[α_, q_, η_, n_, g_] := Module[{H, k, t},
  H = Table[0.0, {i, n}, {j, n}];
  H[[1, n]] = Exp[Sqrt[2 * Pi * α] * g];
  H[[1, 2]] = Exp[-Sqrt[2 * Pi * α] * g];
  H[[1, 1]] = 2 * Cos[2 * Pi * α - I * Sqrt[2 * Pi * α] * q - η];
  For[k = 2, k ≤ n - 1, k = k + 1,
    H[[k, k + 1]] = Exp[-Sqrt[2 * Pi * α] * g];
    H[[k, k]] = 2 * Cos[2 * Pi * k * α + -I * Sqrt[2 * Pi * α] * q - η];
    H[[k, k - 1]] = Exp[Sqrt[2 * Pi * α] * g];];
  H[[n, 1]] = Exp[-Sqrt[2 * Pi * α] * g];
  H[[n, n - 1]] = Exp[Sqrt[2 * Pi * α] * g];
  H[[n, n]] = 2 * Cos[2 * Pi * n * α - I * Sqrt[2 * Pi * α] * q - η];
  Return[H];

```

Figure 4.1: Two mathematica scripts used to find eigenvalues and manipulate data. The script “newnnh” creates the $n \times n$ perturbed Hamiltonian matrix corresponding to the perturbed harper’s equation on a continuous n-site lattice. “hofstadter” creates the Hamiltonian matrix for the ground case (where there is no non-hermitian term).

Where the diagonals have exponential terms as well. We aren’t particularly interested in how the spectrum changes for η , so we can set that parameter to 0. The solution for ϵ is a set of values that ϵ can take on to make the equation hold true. For an $n \times m$ lattice there are n separate eigenvalues. Finding the set of solutions $\{\epsilon\}$ is a problem of finding a linear solution to n differential equations with three terms each (aside from the first and last equations that hold two terms each). Creating a program to do such a task is trivial, yet also uses methods that are computationally expensive. Instead, it is much faster to use Mathematica’s built-in function “Eigenvalues” to find $\{\epsilon\}$ that solve Eq. 4.1.

```

inhh[n_, da_, b_, g_] := Module[{pts, retval, eigtab, A, A1, A2},
  eigtab = ParallelTable[{a, Eigenvalues[newmhh[a, b, 0, n, g]]}, {a, 0, 1, da}];
  pts = Table[{eigtab[[j, 2]][[m]], eigtab[[j, 1]]}, {j, Length[eigtab]}, {m, n}];
  A = Flatten[pts];
  A1 = Table[A[[2 * i - 1]], {i, Length[A] / 2}];
  A2 = Table[A[[2 * i]], {i, Length[A] / 2}];
  pts = Table[{A1[[i]], A2[[i]]}, {i, Length[A2]}];
  Return[pts];]

```

Figure 4.2: Mathematica scripts used to plot eigenvalues. “inhh” computes the eigenvalue spectra that make up the perturbed hofstadter butterfly (in steps $da = d\alpha$). “inhh” has an imaginary counterpart, “imhh”, that performs the same function but only shows the imaginary parts of the resulting butterfly.

```

reimdat[a_, b_, n_, g_] := Module[{eigs, a1, a2, rettab},
  eigs = Eigenvalues[nhh[a, b, 0, n, g]];
  eigs = Sort[eigs];
  a1 = Im[eigs];
  a2 = Re[eigs];
  rettab = Table[{a2[[i]], a1[[i]]}, {i, n}];
  Return[rettab];]

```

Figure 4.3: The code for the script “reimdat”, written in Mathematica. “reimdat” calculates the eigenvalues for the perturbed harper’s equation and returns the list of $\{\text{Im}[\epsilon], \text{Re}[\epsilon]\}$. This data can then be plotted to indicate where an energy eigenvalue has become complex.

Chapter 5

Results

The eigenvalues of Harper's equation, when unperturbed, yield a fractal structure for the eigenvalues as the flux quantum α is varied. For all nonzero values of q , the energy spectrum for Eq. 3.22. is immediately complex for all α values. However, g perturbations produce a transition between real and complex eigenvalues as g increases. This is a novel result for the harper's equation.

5.0.1 Analytic Eigenfunctions

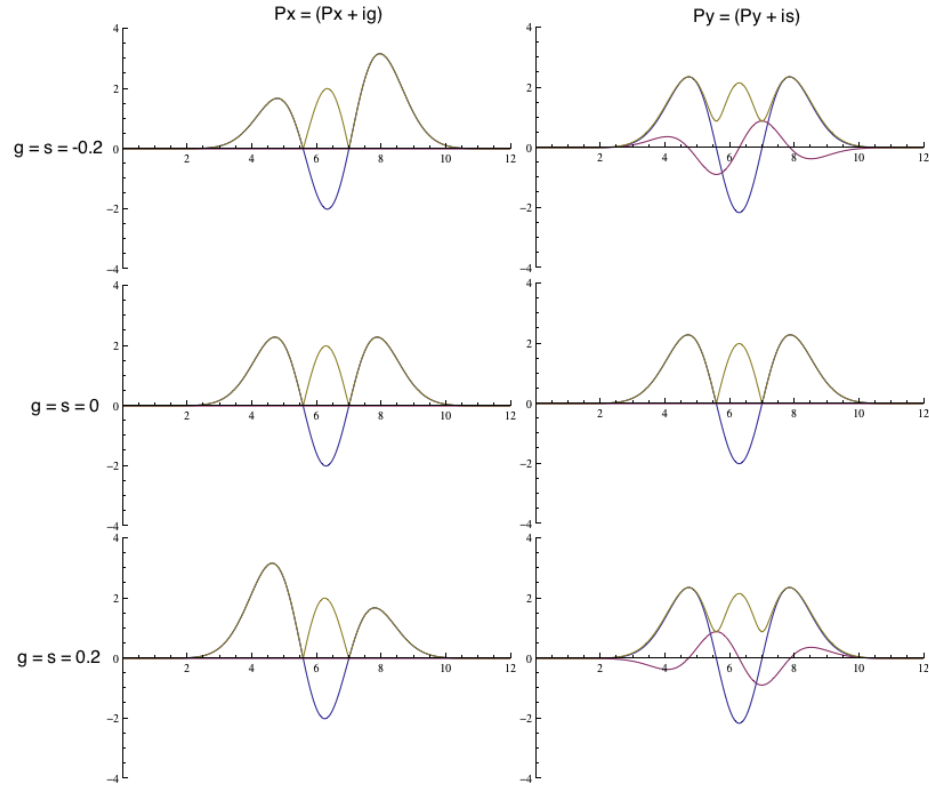


Figure 5.1: Plots of $\phi(x)$ for $n = 2$, and $P \rightarrow (Px + ig)\hat{x} + (Py + iq)\hat{y}$, from Eqs. 3.18. and 3.19.

5.0.2 Hofstadter Butterflies

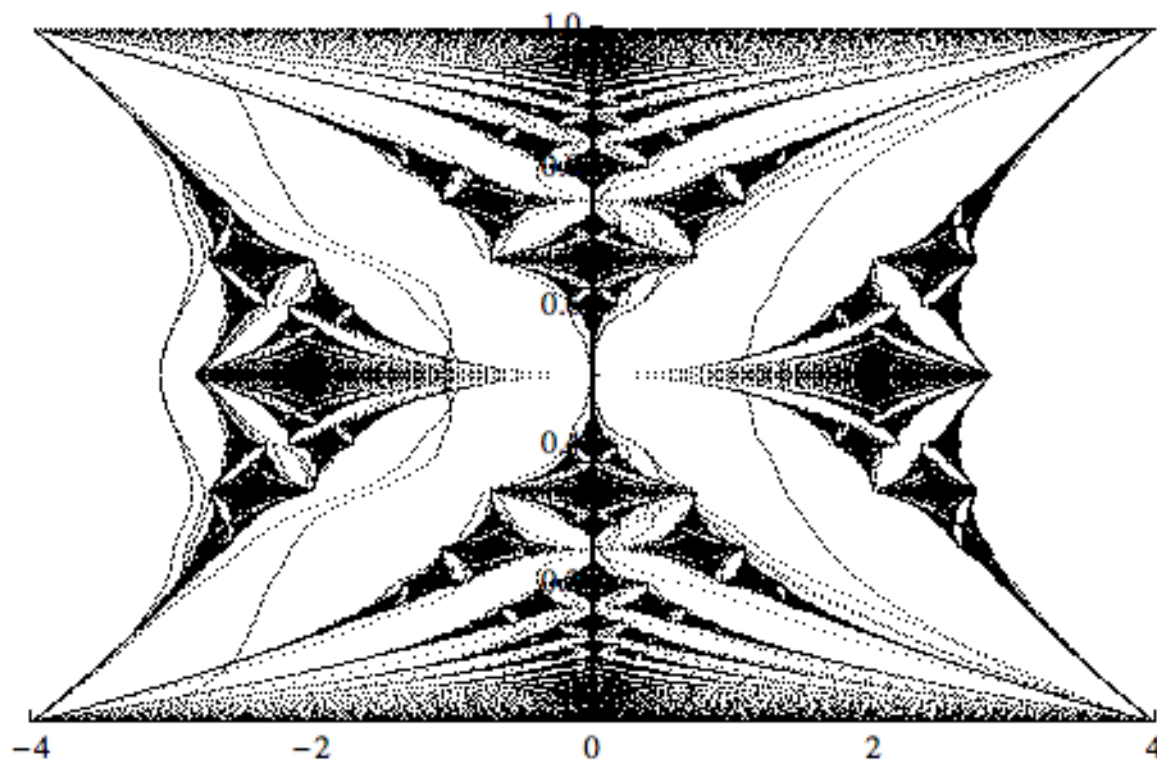


Figure 5.2: The unperturbed Hofstadter butterfly for $n = 200$.

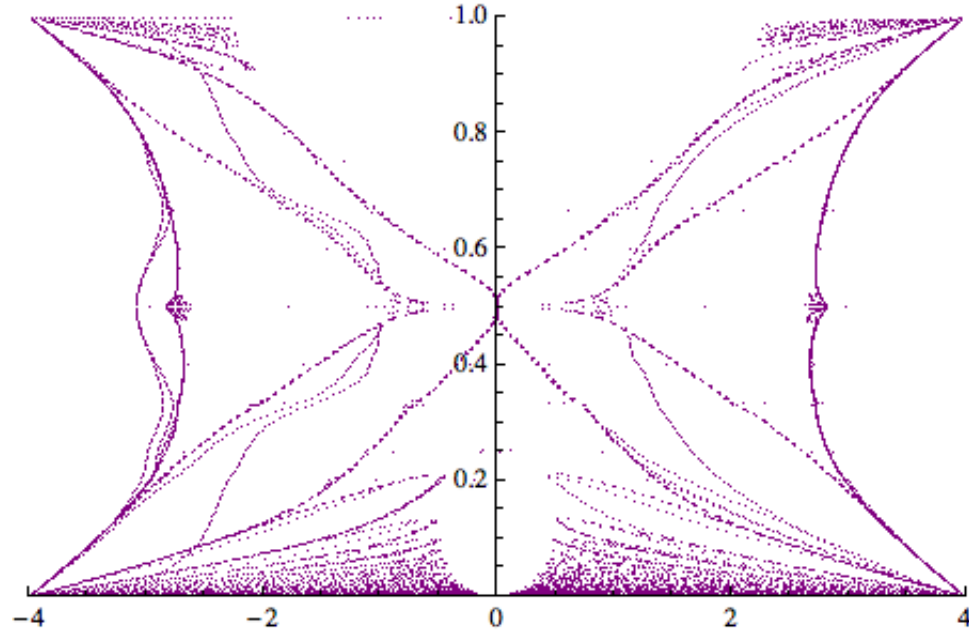


Figure 5.3: The real spectra of the perturbed Hofstadter butterfly for $n = 200$, $g = 0.25$.

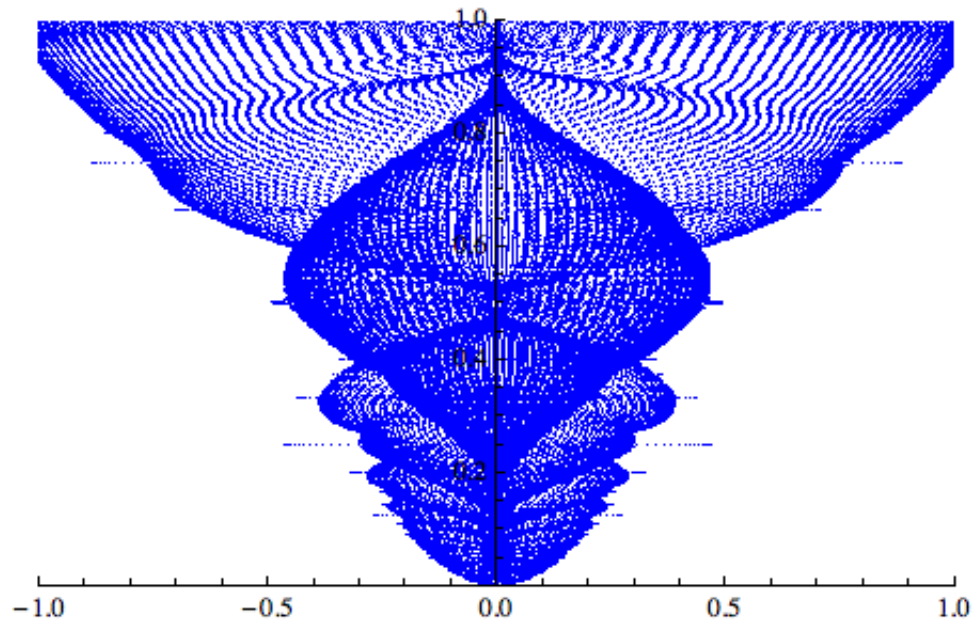


Figure 5.4: The imaginary spectra of the perturbed Hofstadter butterfly for $n = 200$, $g = 0.25$.

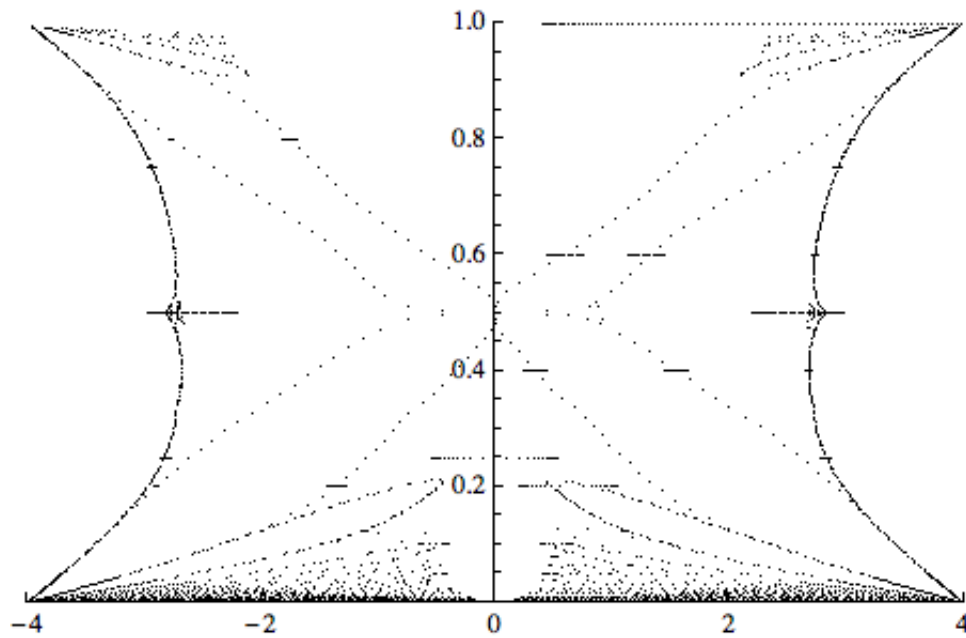


Figure 5.5: The real spectra of the perturbed Hofstadter butterfly for $n = 200$, $q = 0.25$.

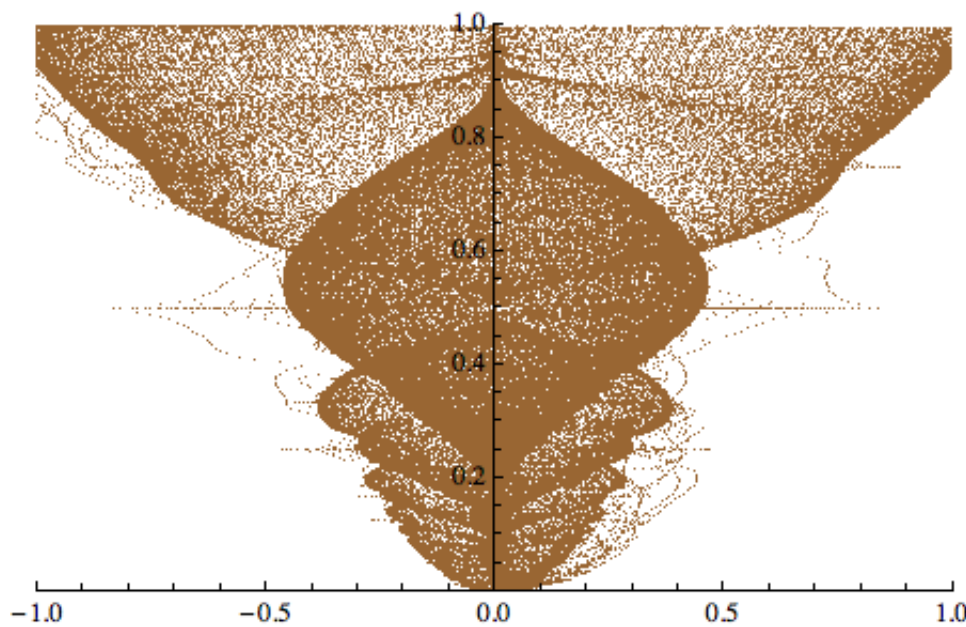


Figure 5.6: The imaginary spectra of the perturbed Hofstadter butterfly for $n = 200$, $q = 0.25$.

5.0.3 Real vs. imaginary Eigenvalues

What follows here are plots of the real vs. imaginary eigenvalue spectrum for given α , q and g parameters, over a 1000 site continuous lattice. Blue graphs represent g perturbations, while orange graphs represent q perturbations. In each plot of each figure, the set of numbers indicates its parameters like so: $\{\alpha, g, q\}$.

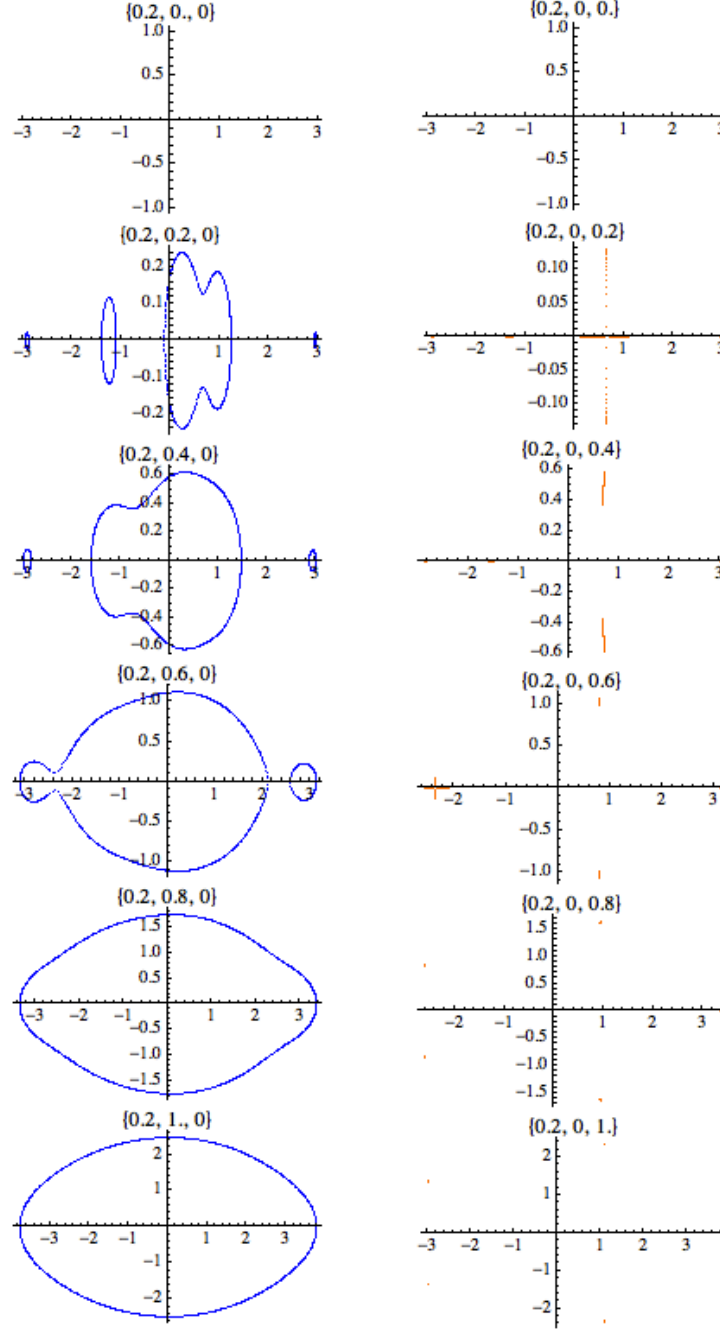


Figure 5.7: Real vs. imaginary eigenvalue plots for $\alpha = 0.2$ on a 1000-site lattice. Blue pertains to g ($\hat{x}$) perturbation and orange to q ($\hat{y}$) perturbation

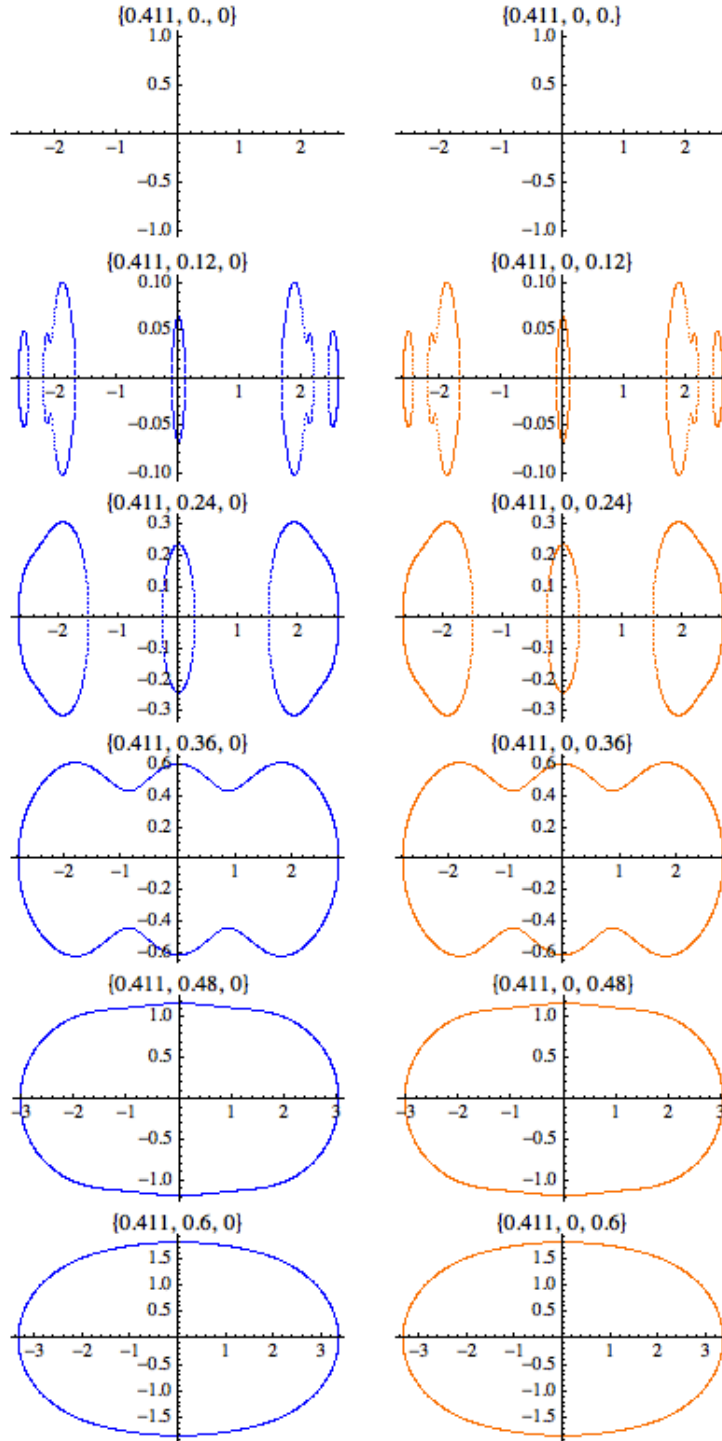


Figure 5.8: Real vs. imaginary eigenvalue plots for $\alpha = 0.411$ on a 1000-site lattice. Blue pertains to $g(\hat{x})$ perturbation and orange to $q(\hat{y})$ perturbation

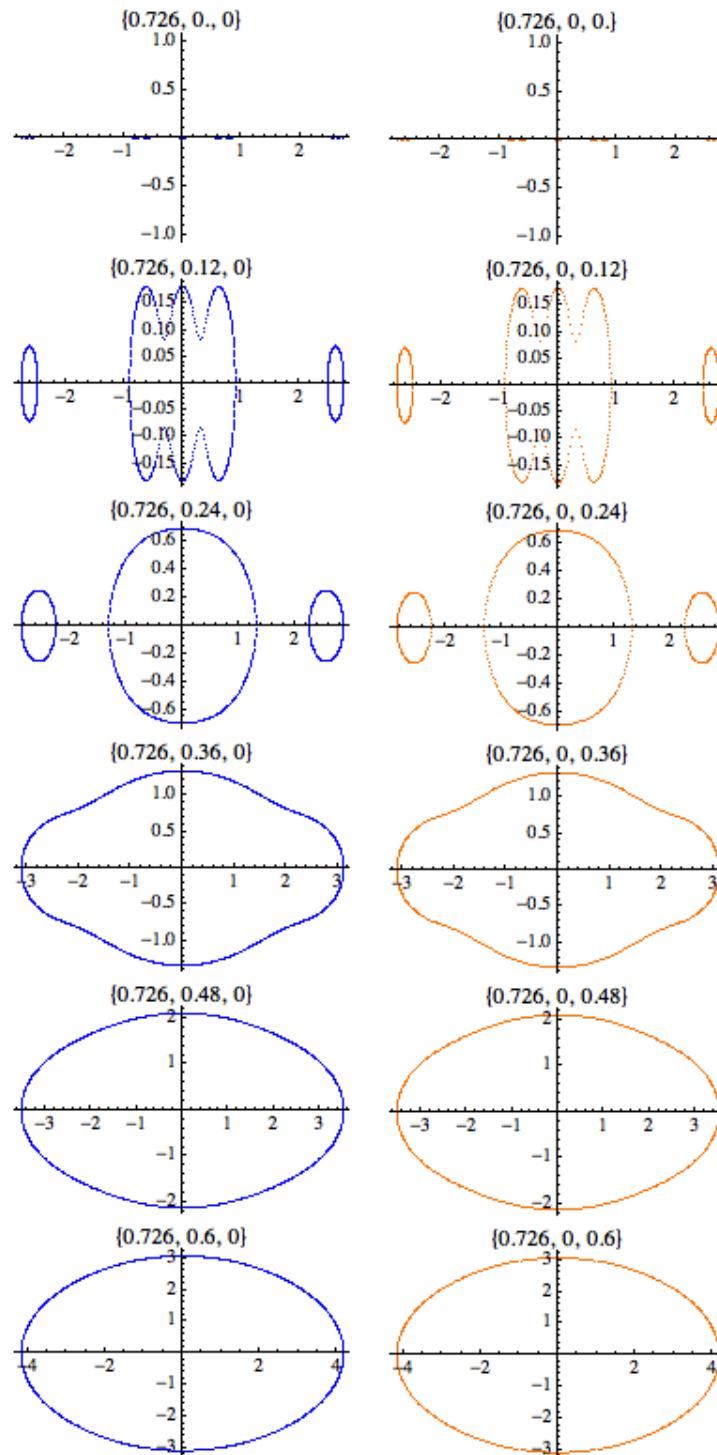


Figure 5.9: Real vs. imaginary eigenvalue plots for $\alpha = 0.726$ on a 1000-site lattice. Blue pertains to $g(\hat{x})$ perturbation and orange to $q(\hat{y})$ perturbation

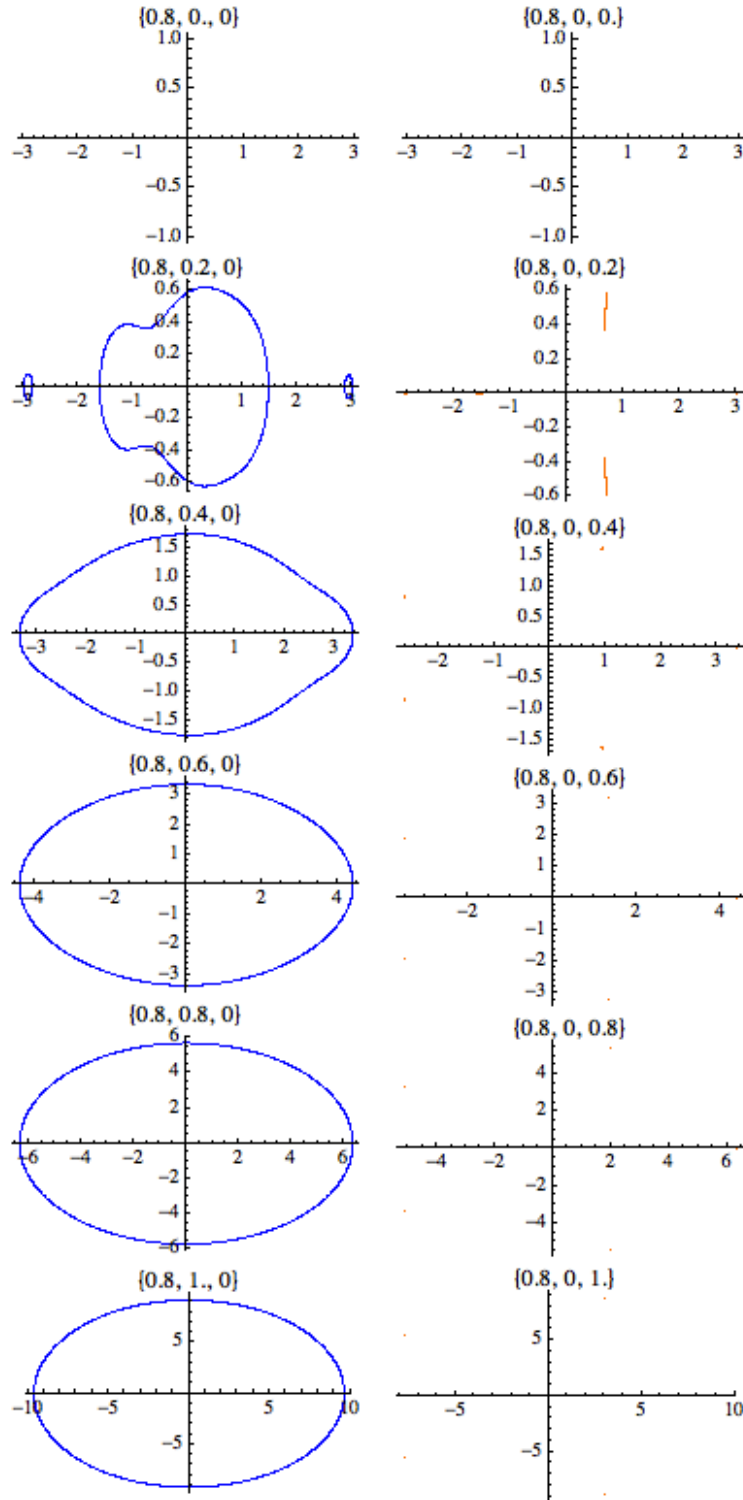


Figure 5.10: Real vs. imaginary eigenvalue plots for $\alpha = 0.8$ on a 1000-site lattice. Blue pertains to $g(\hat{x})$ perturbation and orange to $q(\hat{y})$ perturbation.

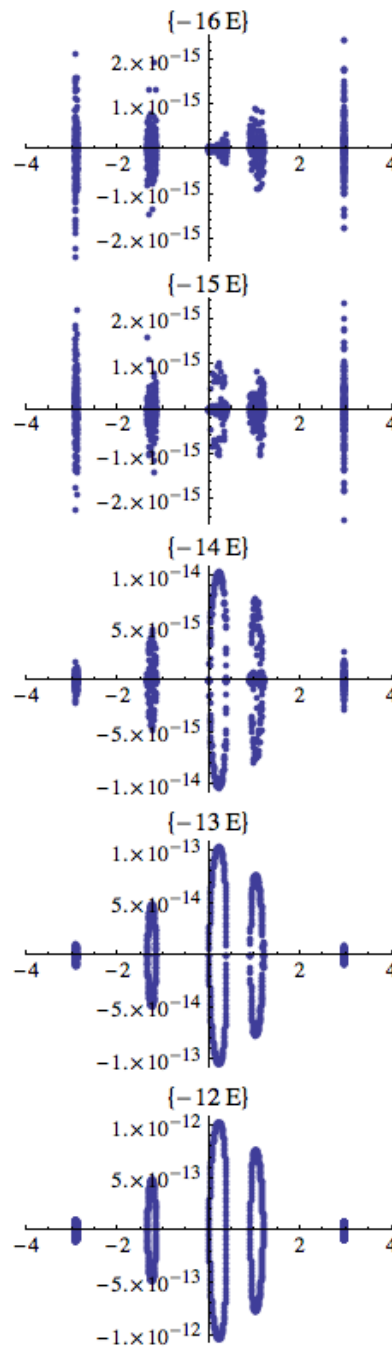


Figure 5.11: Real vs. imaginary eigenvalue plots for $\alpha = 0.2$ on a 1000-site lattice. Blue pertains to g ($\hat{x}$) perturbation. As the g parameter increases on the order of 10^{-15} at $\alpha = 0.2$, the eigenvalue spectrum evolves beyond simple transformations (scaling, shifting, etc.). The bracketed term above each plot indicates the order of the perturbation (e.g. -16E is $g = 10^{-16}$)

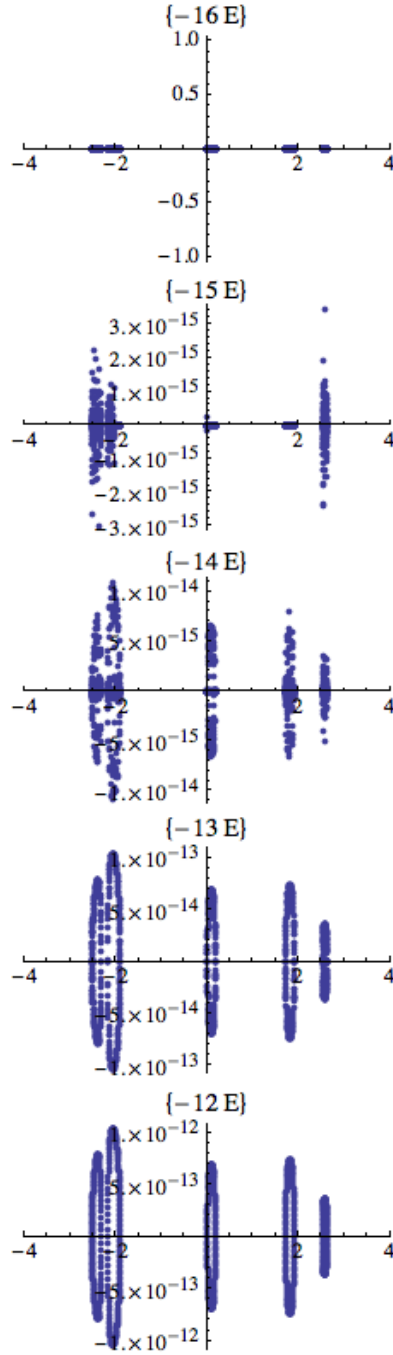


Figure 5.12: Real vs. imaginary eigenvalue plots for $\alpha = 0.4$ on a 1000-site lattice. Blue pertains to $g(\hat{x})$ perturbation. A move from all-real to complex eigenvalues appears while varying g on the order of 10^{-15} at $\alpha = 0.4$. The bracketed term above each plot indicates the order of the perturbation (e.g. -16E is $g = 10^{-16}$)

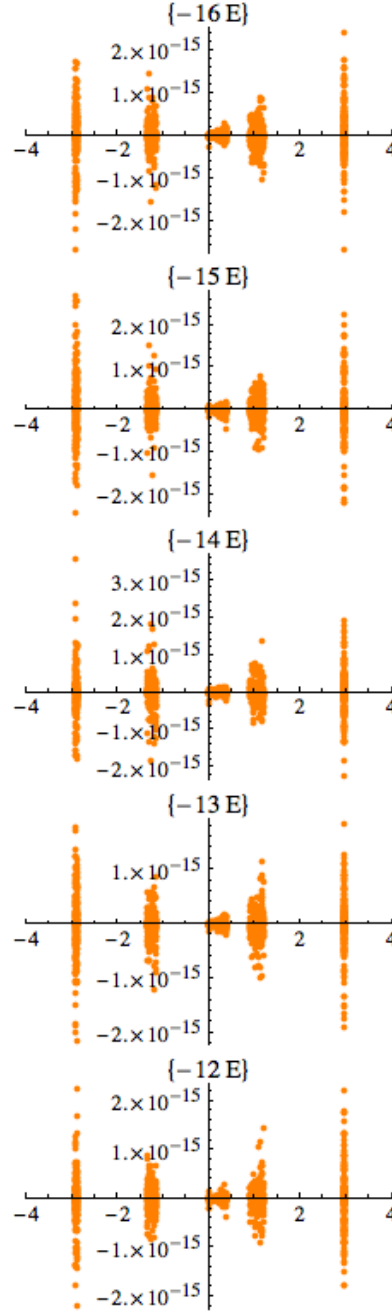


Figure 5.13: Real vs. imaginary eigenvalue plots for $\alpha = 0.2$ on a 1000-site lattice. Orange pertains to q ($\hat{y}$) perturbation. The eigenvalue spectrum scales with q . Despite using the same system parameters as the g perturbation, there appears to be no analog to the g eigenvalue evolution for q on the order of 10^{-15} at $\alpha = 0.2$. The bracketed term above each plot indicates the order of the perturbation (e.g. -16E is $g = 10^{-16}$).

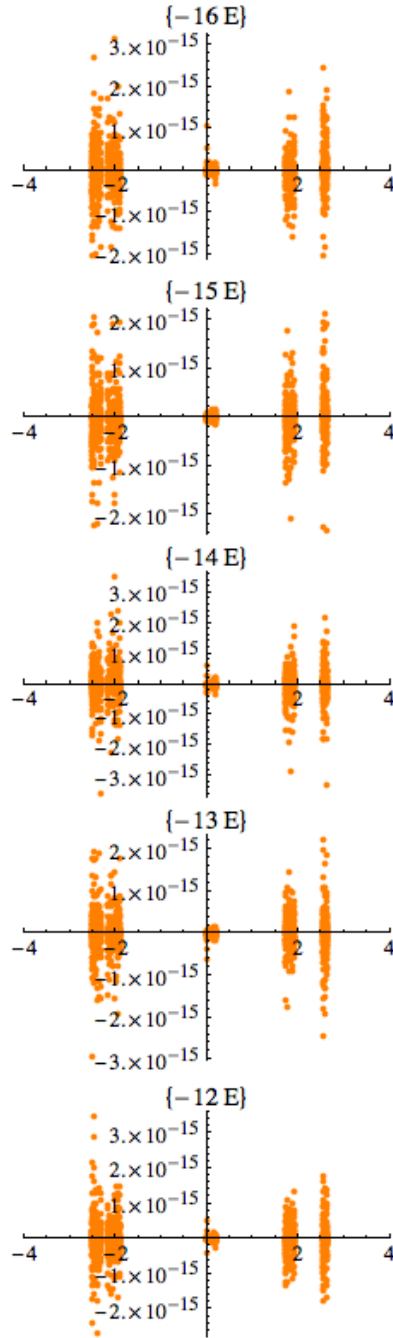


Figure 5.14: Real vs. imaginary eigenvalue plots for $\alpha = 0.4$ on a 1000-site lattice. Orange pertains to q ($\hat{y}$) perturbation. There appears to be no analog to the g eigenvalue evolution for q on the order of 10^{-15} at $\alpha = 0.4$. The eigenvalue spectrum simply scales with q . The bracketed term above each plot indicates the order of the perturbation (e.g. -16E is $g = 10^{-16}$).

Conclusion

As with other systems, the non-hermitian parameters g and q can be interpreted as coupling terms. Further, the qualitative behavior of the analytic eigenvectors may indicate how energy moves within the lattice. With increasing g the analytic eigenvectors skew the wave functions back along the lattice. As shown in Fig 3.5., x -perturbations manifest themselves in the coefficients for the cross terms. With increasing α , the coupling of the $g(m - 1)$ term decreases, and the coupling of the $g(m + 1)$ increases, porting a larger amount of energy forward on the lattice.

It is not surprising, therefore, that there may be no phase transition for the q parameter. For all non-zero q , $\cos[\beta - iq\sqrt{2\pi\alpha}]$ is complex, indicating a complex energy state. The equation considers only the direction perpendicular to this perturbation's directional origins. It is possible that when replaced into an equation without the square lattice ansatz in Eq.1.17, it might exhibit a phase transition. Perhaps, however, the gauge choice pointing the magnetic vector potential in the $\hat{y}$ direction forces any complex perturbation in that direction to create energy eigenvalues. Interestingly, q perturbations seem to exactly copy the behavior of g perturbations for certain α values, exhibiting what may be phase transitions. It also appears, however, that certain values of α completely barr the possibility of a phase transition (Compare Figs. 5.7 and 5.10 with Figs. 5.8 and 5.9.).

The character of the possible phase transition in g is defined by two regions: stable and unstable. Within the stable regime, there exist some values for g such that the resulting energy spectrum is complex, but these values are localized and do not affect the spectrum significantly. For example, the imaginary portions of the eigenvalue spectrum for $\alpha = 0.4$ (Fig. 5.12) continue to be insignificant with respect to the real part of the eigenvalues well after they appear. For the first complex eigenspectrum found for a given set of parameters, the imaginary values tend to be so incredibly small and so fleeting that they can be discounted without too much justification.

According to the Nelson-Hatano literature, where the perturbing parameter becomes sufficiently large, all states, including the ground state, translate into the complex domain. While confirming this fact extends beyond the reach of this document, Fig 5.9. bears the mark that such a phase transition has or may yet occur. Above some critical value for the system, there are only two real eigenvalues for the system. Qualitatively, the middle states have seemed to merge together and ballooned out, no longer crossing the real plane. The imaginary eigenspectrum suddenly surges to the order of the real eigenspectrum, indicating a non-trivial phase transition for the entire system, similar to the features and behavior of the Nelson-Hatano system (Fig. 3.4).

The possibility of a PT-phase transition for the g parameter is novel and well-supported by the Nelson-Hatano model. The only difference between the non-hermitian Hamiltonian for random potentials and the perturbed Harper's equation is the pre-determined potential functions contained in Harper's equation. Furthermore, the system described by the perturbed Harper's equation has its root in the analytic solution of a shifted quantum harmonic oscillator. Since the two coupled oscillator systems outlined in the PT-quantum mechanics section outline an excellent and clean example of a PT-phase transition it would not be a surprise that a system of n -such coupled oscillators would also exhibit a phase transition.

Unlike with the non-hermitian coupled systems examined in the PT quantum mechanics section, the momentum-perturbed quantum Hall system pertains to very complex coupling dynamics whose exact behavior is unclear. A physical example might couple a number of quantum harmonic oscillator systems and observe the dynamics of each pendulum individually to see if some of the oscillators maintain PT-symmetric behavior while others do not. Further, it would look for a critical area or value beyond which the entire system observes a phase transition, a possible physical manifestation of broken PT-symmetry.

Appendix A

Derivations

A.0.4 Hermite Polynomials

Solving the differential equation describing the behavior of a particle in a magnetic field requires a few tricks. Solving Eq. 1.22. means solving a an equation of the form,

$$\phi'' - x^2\phi - 2\epsilon\phi = 0 \quad (\text{A.1})$$

This class of equations can via at least two methods: by assuming that ϕ is some function multiplied by a power series in x , and through a method of ladder operators [23]. I use a combination of the two to find the restriction on ϵ , then to find the final form of ϕ . To begin, notice that for $x \rightarrow \infty$,

$$\phi'' - x^2\phi \approx 0 \quad (\text{A.2})$$

Suggesting a solution of the form,

$$\phi = u[x]e^{-x^2/2} \quad (\text{A.3})$$

Applying this ansatz in A.2. eventually becomes,

$$e^{-\frac{x^2}{2}} ((2\epsilon - 1)u[x] + u''[x] - 2xu'[x]) = 0 \quad (\text{A.4})$$

Now, assume that u is a power series in x ,

$$u = \sum_{i=0}^{\infty} \alpha_i x^i. \quad (\text{A.5})$$

The expression from A.4 becomes

$$\begin{aligned} (2\epsilon - 1) \sum_{i=0}^{\infty} \alpha_i x^i + \sum_{i=0}^{\infty} (i-1) i \alpha_i x^{i-2} - 2x \sum_{i=0}^{\infty} i \alpha_i x^{i-1} &= 0, \\ \Rightarrow (2\epsilon - 1) \sum_{i=0}^{\infty} \alpha_i x^i + \sum_{i=0}^{\infty} (i-1) i \alpha_i x^{i-2} - 2 \sum_{i=0}^{\infty} i \alpha_i x^i &= 0, \end{aligned}$$

$$\Rightarrow \sum_{i=0}^{\infty} (2\epsilon - 1 - 2i)a_i x^i + \sum_{i=0}^{\infty} (i-1)ia_i x^{i-2} = 0.$$

Now, there is some lowest coefficient such that $a_l = 0$. The first two terms of the x^{i-2} sum should therefore be 0, so the iterator can therefore be changed $i \rightarrow i+2$ in the right sum, while still starting from $i = 0$. Updating the iterator yields,

$$\sum_{i=0}^{\infty} (2\epsilon - 1 - 2i)a_i x^i - \sum_{i=0}^{\infty} (i+1)(i+2)a_{i+2} x^i = 0$$

We can now bring both terms inside the summation, finally yielding the equality,

$$(i+1)(i+2)a_{i+2} = (2\epsilon - 1 - 2i)a_i$$

This equation diverges, which represents a non-physical solution (infinite wave function). Therefore, we ask that the wave function have some finite level n such that

$$a_{n+2} = \frac{2\epsilon - 1 - 2n}{(n+1)(n+2)} a_n = 0 \quad (\text{A.6})$$

This can be done by making the numerator zero,

$$2\epsilon - 1 - 2n = 0. \quad (\text{A.7})$$

The condition implies that ϵ take the following value,

$$\epsilon = n + 1/2. \quad (\text{A.8})$$

Thus, the energy levels are quantized for a particle in a magnetic field. Having found the condition of energy quantization, we use the ladder operator method to generate wave functions easily (see [23] for a thorough treatment of the ladder operator method). Rewriting Eq. 1.22. with the constraint in A.8,

$$\phi'' - x^2 \phi = (2n+1)\phi. \quad (\text{A.9})$$

We can rewrite this equation in terms of the derivative operators:

$$D = \frac{\partial}{\partial x}.$$

Now, A.9 can be factored into two equations involving D ,

$$(D-x)(D+x)\phi_n = -2n\phi_n, \quad (\text{A.10})$$

And,

$$(D+x)(D-x)\phi_n = -2(n+1)\phi_n. \quad (\text{A.11})$$

To see how the operators $(D+x)$ and $(D-x)$ affect the wave function, consider what happens to the differential equation for some other wave function ϕ_m when acted upon by $(D+x)$ and $(D-x)$. From A.10,

$$(D+x)(D-x)(D+x)\phi_m = -2m(D+x)\phi_m. \quad (\text{A.12})$$

From A.11,

$$(D - x)(D + x)(D - x)\phi_m = -2(m + 1)(D - x)\phi_m. \quad (\text{A.13})$$

Compare A.10 and A.12. The equations are identical for, $y_n = [(D - x)y_m]$ and $n = m + 1$, implying,

$$y_{m+1} = (D - x)y_m. \quad (\text{A.14})$$

Similarly, for A.10 and A.12, $y_n = [(D + x)y_m]$, and $n = m - 1$,

$$y_{m-1} = (D + x)y_m. \quad (\text{A.15})$$

These equations are called ladder operators. They take any y_i to y_{i+1} or y_{i-1} except where there is no higher or lower rung. For instance, there should be some lowest function ϕ_0 such that,

$$(D + x)\phi_0 = 0. \quad (\text{A.16})$$

Solving this equation yields,

$$\phi_0 = e^{-\frac{x^2}{2}}. \quad (\text{A.17})$$

With this equation, in conjunction with A.14., any wave function can be created. ϕ_1 for example:

$$\phi_1 = (D - x)\phi_0 = A(D - x)e^{-\frac{x^2}{2}}.$$

The equation for some ϕ_i , for general i , is therefore,

$$\phi_i = A(D - x)^i e^{-\frac{x^2}{2}}. \quad (\text{A.18})$$

Or, in its equivalent form,

$$\phi_n = Ae^{-\frac{x^2}{2}} H_n(X) = A(-1)^n e^{-\frac{x^2}{2}} e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2}. \quad (\text{A.19})$$

$H_n(x)$ are the Hermite polynomials. Now that we have the full form of the solution, we need to orthonormalize it. The Hermite polynomials are already orthogonal, meaning,

$$\int_{-\infty}^{\infty} H_n[x]H_m[x]dx = \begin{cases} 0 & m \neq n \\ 1 & n = m \end{cases}. \quad (\text{A.20})$$

With the Hermite polynomials are already orthogonal, the full solution is orthogonal as well. However, the full solution still needs to be normalized over $x \in [\infty, \infty]$. For normalization of each wave function, we want,

$$1 = \int_{-\infty}^{\infty} |\phi^* \phi| dx$$

The conjugate of some ϕ_n is simply ϕ_n ,

$$1 = A \int_{-\infty}^{\infty} e^{-x^2} H_n[x]H_n[x]dx. \quad (\text{A.21})$$

Expanding the first Hermite polynomial yields,

$$1 = A^2(-1)^n \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n} [e^{-x^2}] H_n[x] dx. \quad (\text{A.22})$$

Integrating by parts, we have,

$$1 = -A^2(-1)^{n-1} \left(\left[\frac{\partial^n}{\partial x^n} [e^{-x^2}] H_n[x] \right]_{-\infty}^{\infty} - 2 \int_{-\infty}^{\infty} \frac{\partial^{n-1}}{\partial x^{n-1}} [e^{-x^2}] H_n[x] dx \right). \quad (\text{A.23})$$

The left part of the equation is 0 at its endpoints, leaving the integral on the right which can be further integrated by parts until in the same manner until the $H_0(x)$ polynomial, which is 1. Furthermore, there is a factor of $(-1)^n$ from these integrations by parts, since each integral has a negative sign. This factor combines with the leading $(-1)^n$ to make $(-1)^{2n} = 1$. After these steps, we are left with,

$$1 = A^2 \int_{-\infty}^{\infty} e^{-x^2} H_n[x] dx. \quad (\text{A.24})$$

Integrating by parts yet again,

$$1 = A^2 \left(\left[e^{-x^2} H_n(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-x^2} H'_n[x] dx \right). \quad (\text{A.25})$$

Yet again, the left side is 0. Now, using the relation $H'_n(x) = 2nH_{n-1}(x)$ [23],

$$1 = A^2 2n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x) dx, \quad (\text{A.26})$$

$$1 = A^2 2^2 n(n-1) \int_{-\infty}^{\infty} e^{-x^2} H_{n-2}(x) dx,$$

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$$1 = A^2 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx = A^2 2^n n! \sqrt{\pi} \quad (\text{A.27})$$

Finally, we have the constant A to normalize the wavefunction ϕ_n :

$$A = \frac{1}{\sqrt{2^n n! \pi^{1/2}}}. \quad (\text{A.28})$$

Finally, from A.19 and A.28, comes the form of ϕ_n ,

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \pi^{1/2}}} e^{-x^2/2} H_n(x). \quad (\text{A.29})$$

A.0.5 Longitudinal Current[4]

Beginning from

$$I_y = \int J(r) \cdot \hat{x} dy, \quad (\text{A.30})$$

$$I_y = \frac{1}{L} \int \int J_y(x, y) dx dy, \quad (\text{A.31})$$

$$I_y = -\frac{e}{L} \int d^2 p v_y, \quad (\text{A.32})$$

$$I_y = -\frac{e}{L} \sum_j v_y. \quad (\text{A.33})$$

From Eq. 1.11. get the form of v_y , making the current expression,

$$I_y = -\frac{e}{2\pi\hbar} \int dp_y v_{p_y} \hat{n}_p \quad (\text{A.34})$$

Which is equivalent to Eq. 1.35.

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