for $x_1 \leqq x \leqq x_2$, with $(x, y)$ in a neighborhood of $C_0$, and $y - y_0(x)$ and $y' - y_0'(x)$ not both zero, then $C_0$ furnishes a proper (or improper) strong relative minimum for $J = \int_{x_1}^{x_2} f(x, y, y')dx$, $G(x, y, \bar{y}, y', \bar{y}') \equiv f(x, y, y') - f(x, \bar{y}, \bar{y}') - (y - \bar{y})f_{\bar{y}}(x, \bar{y}, \bar{y}') - (y' - \bar{y}')f_{\bar{y}'}(x, \bar{y}, \bar{y}')$. Examples were given which satisfy these sufficient conditions but which do not yield to classical methods.

3. The finite abstract groups in which every subgroup is invariant (Hamiltonian groups) are well known, as are also the groups in which every subgroup is abelian. Classifying the finite abstract groups according to the number of complete sets of non-invariant conjugate subgroups contained in them, Professor Sigley discussed the class of groups which contain a single complete set of non-invariant conjugate subgroups. Necessary and sufficient conditions that a group belong to this class were obtained.

4. It was shown by Professor Finkel how the derivatives of $a^x$ and $\sin^{-1} x$ may be derived directly by the $\Delta$-process, without reverting to the derivatives of the logarithmic and sine functions.

5. Mr. Gillam sketched a development of three-dimensional analytic euclidean geometry (free of coördinates) with "point" as the single undefined element and "distance" as the only primitive relation. Lines and planes were defined as determinantal loci.

6. Mr. Wulftange presented a new method for the numerical solution of quadratic congruences which is believed to possess certain practical advantages.

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## THE TRANSCENDENCE OF $\pi$

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Among the proofs of the transcendence of $e$, which are in general variations and simplifications of the original proof of Hermite, perhaps the simplest is that of A. Hurwitz.† His solution of the problem contains an ingenious device which we now employ to obtain a relatively simple proof of the transcendence of $\pi$.

We assume that $\pi$ is an algebraic number, and show that this leads to a contradiction. Since the product of two algebraic numbers is an algebraic number, the quantity $i\pi$ is a root of an algebraic equation with integral coefficients

$$\theta_1(x) = 0, \tag{1}$$

whose roots are $\alpha_1 = i\pi, \alpha_2, \alpha_3, \cdots, \alpha_n$. Using Euler's relation $e^{i\pi} + 1 = 0$, we have

$$(e^{\alpha_1} + 1)(e^{\alpha_2} + 1) \cdots (e^{\alpha_n} + 1) = 0. \tag{2}$$

We now construct an algebraic equation with integral coefficients whose roots are the exponents in the expansion of (2). First consider the exponents

$$\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \cdots, \alpha_{n-1} + \alpha_n. \tag{3}$$

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† A. Hurwitz, Beweis der Transendenz der Zahl $e$, Mathematische Annalen, vol. 43, 1893, pp. 220–221 (also in his Mathematische Werke, vol. 2, pp. 134–135).

By equation (1), the elementary symmetric functions of $\alpha_1, \alpha_2, \cdots, \alpha_n$ are rational numbers. Hence the elementary symmetric functions of the quantities (3) are rational numbers. It follows that the quantities (3) are roots of

$$\theta_2(x) = 0, \tag{4}$$

an algebraic equation with integral coefficients. Similarly, the sums of the $\alpha$'s taken three at a time are the ${}_nC_3$ roots of

$$\theta_3(x) = 0. \tag{5}$$

Proceeding thus, we obtain

$$\theta_4(x) = 0, \theta_5(x) = 0, \cdots, \theta_n(x) = 0, \tag{6}$$

algebraic equations with integral coefficients, whose roots are the sums of the $\alpha$'s taken $4, 5, \cdots, n$ at a time respectively. The product equation

$$\theta_1(x)\theta_2(x) \cdots \theta_n(x) = 0 \tag{7}$$

has roots which are precisely the exponents in the expansion of (2).

The deletion of zero roots (if any) from equation (7) gives

$$\theta(x) = cx^r + c_1x^{r-1} + \cdots + c_r = 0, \tag{8}$$

whose roots $\beta_1, \beta_2, \cdots, \beta_r$ are the non-vanishing exponents in the expansion of (2), and whose coefficients are integers. Hence (2) may be written in the form

$$e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_r} + k = 0, \tag{9}$$

where $k$ is a positive integer.

We define

$$f(x) = \frac{c^s x^{p-1}\{\theta(x)\}^p}{(p-1)!}, \tag{10}$$

where $s = rp - 1$, and $p$ is a prime to be specified. Also we define

$$F(x) = f(x) + f^{(1)}(x) + f^{(2)}(x) + \cdots + f^{(s+p+1)}(x), \tag{11}$$

noting, with thanks to Hurwitz, that the derivative of $e^{-x}F(x)$ is $-e^{-x}f(x)$. Hence we may write

$$e^{-x}F(x) - e^0F(0) = \int_0^x - e^{-\xi}f(\xi)d\xi.$$

The substitution $\xi = \tau x$ produces

$$F(x) - e^xF(0) = -x\int_0^1 e^{(1-\tau)x}f(\tau x)d\tau.$$

Let $x$ range over the values $\beta_1, \beta_2, \cdots, \beta_r$ and add the resulting equations. Using (9). we obtain

$$\sum_{j=1}^{r} F(\beta_j) + kF(0) = -\sum_{j=1}^{r} \beta_j \int_0^1 e^{(1-\tau)\beta_j} f(\tau\beta_j) d\tau. \tag{12}$$

This result gives us the contradiction we desire. For we shall choose the prime $p$ to make the left side a non-zero integer, and the right side as small as we please.

By (10), we have

$$\sum_{j=1}^{r} f^{(t)}(\beta_j) = 0, \quad \text{for} \quad 0 \leqq t < p.$$

Also by (10) the polynomial obtained by multiplying $f(x)$ by $(p-1)!$ has integral coefficients. Since the product of $p$ consecutive positive integers is divisible by $p!$, the $p$th and higher derivatives of $(p-1)!f(x)$ are polynomials in $x$ with integral coefficients divisible by $p!$. Hence the $p$th and higher derivatives of $f(x)$ are polynomials with integral coefficients each of which is divisible by $p$. That each of these coefficients is also divisible by $c^s$ is obvious from the definition (10). Thus we have shown that, for $t \geqq p$, the quantity $f^{(t)}(\beta_j)$ is a polynomial in $\beta_j$ of degree at most $s$, each of whose coefficients is divisible by $pc^s$. By (8), a symmetric function of $\beta_1, \beta_2, \cdots, \beta_r$ with integral coefficients and of degree at most $s$ is an integer provided each coefficient is divisible by $c^s$ (by the fundamental theorem on symmetric functions). Hence

$$\sum_{j=1}^{r} f^{(t)}(\beta_j) = pk_t, \qquad (t = p, p+1, \cdots, p+s),$$

where the $k_t$ are integers. It follows that

$$\sum_{j=1}^{r} F(\beta_j) = p \sum_{t=p}^{p+s} k_t.$$

In order to complete the proof that the left side of (12) is a non-zero integer, we now show that $kF(0)$ is an integer prime to $p$. From (10) it is clear that

$$\begin{aligned} f^{(t)}(0) &= 0, && (t = 0, 1, \cdots, p-2), \\ f^{(p-1)}(0) &= c^s c_r{}^p, \\ f^{(t)}(0) &= pK_t, && (t = p, p+1, \cdots, p+s), \end{aligned}$$

where the $K_t$ are integers. If $p$ is chosen greater than each of $k$, $c$, $c_r$ (possible since the number of primes is infinite), the desired result follows from (11).

Finally, the right side of (12) equals

$$-\sum_{j=1}^{r} \frac{1}{c} \int_0^1 \frac{\{c^r \beta_j \theta(\tau\beta_j)\}^p}{(p-1)!} e^{(1-\tau)\beta_j} d\tau.$$

This is a finite sum, each term of which may be made as small as we wish by choosing $p$ very large, because

$$\lim_{p\to\infty} \frac{\{c^r \beta_j \theta(\tau\beta_j)\}^p}{(p-1)!} = 0.$$