A Proof that Euler Missed: Evaluating $\zeta(2)$ the Easy Way

Tom M. Apostol

R. Apéry [1] was the first to prove the irrationality of

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Motivated by Apéry's proof, F. Beukers [2] has given a shorter proof which uses multiple integrals to establish the irrationality of both $\zeta(2)$ and $\zeta(3)$. In this note we show that one of the double integrals considered by Beukers,

$$I = \int_0^1 \int_0^1 \frac{1}{1 - xy}\, dxdy,$$

can be used to establish directly that $\zeta(2) = \pi^2/6$. This evaluation has been presented by the author for a number of years in elementary calculus courses, but does not seem to be recorded in the literature.

The relation between the foregoing integral and $\zeta(2)$ is obtained by expanding the integrand in a geometric series and integrating term by term. Thus, we have

$$I = \int_0^1 \int_0^1 \sum_{n=0}^{\infty} x^n y^n \, dxdy =$$

$$\int_0^1 \sum_{n=0}^{\infty} \frac{y^n}{n+1}\, dy = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \zeta(2).$$

Now we evaluate the integral another way and show that $I = \pi^2/6$. We simply rotate the coordinate axes clockwise through an angle of $\pi/4$ radians by introducing the change of variables

$$x = \frac{u - v}{\sqrt{2}}, \quad y = \frac{u + v}{\sqrt{2}}$$

so that $1 - xy = (2 - u^2 + v^2)/2$. The new region of integration in the uv-plane is a square with two opposite vertices at $(0,0)$ and $(\sqrt{2},0)$. Making use of the symmetry of this square about the u-axis we find

$$I = 4 \int_0^{1/\sqrt{2}} \left(\int_0^u \frac{dv}{2 - u^2 + v^2} \right) du$$

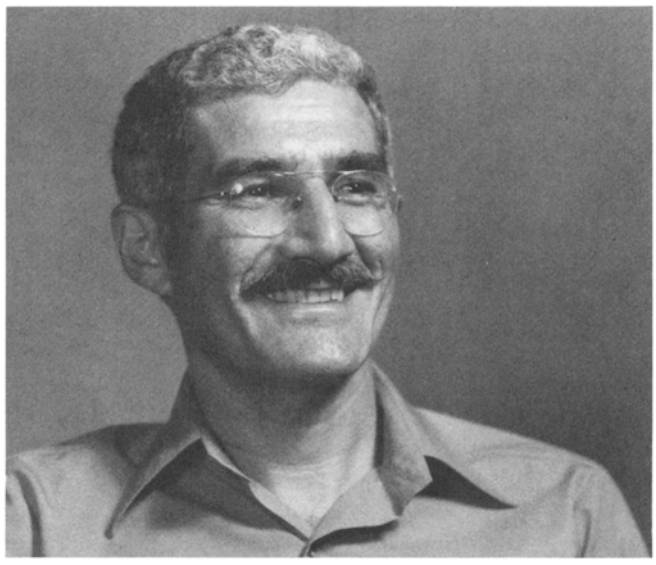
$$+ 4 \int_{1/\sqrt{2}}^{\sqrt{2}} \left(\int_0^{\sqrt{2}-u} \frac{dv}{2 - u^2 + v^2} \right) du.$$

Since

$$\int_0^x \frac{dt}{a^2 + t^2} = \frac{1}{a} \arctan \frac{x}{a}$$

we have

$$\int_0^u \frac{dv}{2 - u^2 + v^2} = \frac{1}{\sqrt{2 - u^2}} \arctan \frac{u}{\sqrt{2 - u^2}}$$



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and

$$\int_0^{\sqrt{2}-u} \frac{dv}{2-u^2+v^2} = \frac{1}{\sqrt{2-u^2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}}$$

hence

$$I = 4\int_0^{1/\sqrt{2}} \arctan \frac{u}{\sqrt{2-u^2}} \frac{du}{\sqrt{2-u^2}}$$

$$+ 4\int_{1/\sqrt{2}}^{\sqrt{2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}} \frac{du}{\sqrt{2-u^2}} = I_1 + I_2,$$

say. Put $u = \sqrt{2}\sin\theta$ in I_1 so that $du = \sqrt{2}\cos\theta\, d\theta = \sqrt{2-u^2}\, d\theta$, and $\tan\theta = u/\sqrt{2-u^2}$. This gives us

$$I_1 = 4\int_0^{\pi/6} \theta\, d\theta = 2\left(\frac{\pi}{6}\right)^2.$$

In I_2 we put $u = \sqrt{2}\cos 2\theta$ so that

$$du = -2\sqrt{2}\sin 2\theta\, d\theta = -2\sqrt{2}\sqrt{1-\cos^2 2\theta}\, d\theta = -2\sqrt{2}\sqrt{1-u^2/2}\, d\theta = -2\sqrt{2-u^2}\, d\theta,$$

and

$$\frac{\sqrt{2}-u}{\sqrt{2-u^2}} = \frac{\sqrt{2}(1-\cos 2\theta)}{\sqrt{2-2\cos^2 2\theta}}$$

$$= \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan\theta,$$

hence

$$I_2 = 8\int_0^{\pi/6} \theta\, d\theta = 4\left(\frac{\pi}{6}\right)^2.$$

Therefore $I = I_1 + I_2 = 6\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}$.

Note. Another evaluation of $\zeta(2)$ using double integrals in a less straightforward manner was given by F. Goldscheider [3] in response to a problem proposed by P. Stäckel. He considers the two double integrals

$$P = \int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} \text{ and } Q = \int_0^1 \int_0^1 \frac{dx\,dy}{1+xy}$$

and shows first that $P - Q = \frac{1}{2}P$ so $P = 2Q$. On the other hand,

$$P + Q = \int_{-1}^1 dy \int_0^1 \frac{dx}{1+xy}.$$

The substitution $u = y + \frac{1}{2}x(y^2-1)$ in the integral with respect to y converts this to

$$P + Q = \int_{-1}^1 du \int_0^1 \frac{dx}{1+2ux+x^2}.$$

Now put $u = \cos\varphi$ so that $(\sin\varphi)/(1+2ux+x^2) = \frac{d}{dx}\left(\arctan\frac{x+\cos\varphi}{\sin\varphi}\right)$, hence

$$P + Q = \frac{1}{2}\int_0^{\pi} \varphi\, d\varphi = \frac{\pi^2}{4},$$

which, together with $P = 2Q$, implies $P = \pi^2/6$.

References

1. R. Apéry (1979) Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque 61*:11–13. Paris: Societé Mathématique de France.
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