ON THE SERIES OF PRIME RECIPROCALS

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Let p_n be the nth prime. We give another proof of the

THEOREM. *The series* $\sum_{n=1}^{\infty} (1/p_n)$ *diverges.*

PROOF. Assume the contrary, and fix k so that

$$\sum_{n=k+1}^{\infty} (1/p_n) < 1/2. \tag{1}$$

Let $Q = p_1 p_2 \cdots p_k$.

We consider now the sum $S(r) = \sum_{i=1}^{r} [1/(1+iQ)]$, where r is any positive integer. Since $1+iQ$ is prime to Q, all the prime factors of all these denominators are from a finite segment of primes which we call $P(r)$:

$$P(r) = \{p_{k+1}, p_{k+2}, \cdots, p_{m(r)}\}.$$

Now let $S(r, j)$ stand for the sum of those terms in the sum $S(r)$ whose denominators $1+iQ$ have just j (not assumed distinct) prime factors. Each such term has the form $1/q_1 q_2 \cdots q_j$, with each $q_t \in P(r)$. But every such term occurs at least once in the expansion of $[\sum_{n=k+1}^{m(r)} (1/p_n)]^j$, so by (1) $S(r, j) < 1/2^j$. Thus for each r,

$$S(r) = \sum_j S(r, j) < \sum_j (1/2^j) < 1.$$

So $\sum_{i=1}^{\infty} [1/(1+iQ)]$ converges, which in turn implies that the harmonic series does.

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