A Very Short Proof of the Infinitude of Primes

If we suppose that the number of primes is not infinite, then by using an "even-odd" argument, we prove that 3 is a greatest prime, a contradiction.

Theorem. *(Euclid [**1**], [**3**, p. 3]) There are infinitely many primes.*

Euclid's famous theorem on the infinitude of primes has fascinated generations of mathematicians since its first and famous demonstration given by Euclid (300 B.C.) in his celebrated *Elements* ([**1**] Book IX, Proposition 20). Even after almost two and a half millennia, Euclid's theorem on the infinitude of primes stands as an excellent model of reasoning. In [**2**] the author of this note provided a comprehensive historical survey of different proofs of Euclid's theorem on the infinitude of primes. In this note, we give a short elementary proof of Euclid's theorem as follows.

Proof. Suppose that $p_1 = 2 < p_2 = 3 < p_3 < \cdots < p_k$ are all the primes. Consider the product $P = 3p_3p_4 \cdots p_k$. Then by the assumption it follows that $\{1, 2, 2^2, \ldots, 2^n, \ldots\}$ is a set of all positive integers that are relatively prime to P. In particular, since 2 is relatively prime to P, it follows that $P - 2$ is also relatively prime to P. Therefore, $P - 2$ is an odd integer belonging to the set $\{1, 2, 2^2, \ldots, 2^n, \ldots\}$. This shows that $P - 2 = 1$, i.e., $P = 3$. This means that 3 is a greatest prime number. A contradiction, and the proof is completed. ■

Remark. Notice that our proof works if we replace the prime $p_1 = 2$ by any prime p_j with $j \in \{2, 3, \ldots, k-1\}$. Then assuming the analogous argument as in the proof above, we arrive at the equation $p_1p_2 \cdots p_{j-1}p_{j+1} \cdots p_k - p_j = 1$, which is impossible because the left-hand side of the previous equation is greater than $2p_k - p_j > p_j > 1$. Observe also that this argument cannot be applied for the "greatest prime" p_k instead of p_j; namely, for $j = k$ we find that $p_1p_2 \cdots p_{k-1} = p_k + 1$. However, we can see from several number theory arguments that the previous equation is satisfied only for $k = 3$ (that is, $2 \cdot 3 = 5 + 1$).

REFERENCES

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2. R. Meštrović, Euclid's theorem on the infinitude of primes: A historical survey of its proofs (300 B.C.–2012), 66 pages, preprint, http://www.arXiv:1202.3670v2[math.HO], 2012.
3. P. Ribenboim, *The Little Book of Bigger Primes*. Second ed. Springer-Verlag, New York, 2004.

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