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THE STANDARD DOUBLE SOAP BUBBLE IN $\mathbf{R}^2$ UNIQUELY MINIMIZES PERIMETER

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Of course the circle is the least-perimeter way to enclose a region of prescribed area in the plane. This paper proves that a certain standard "double bubble" is the least-perimeter way to enclose and separate two regions of prescribed areas. The solution for three regions remains conjectural.

1. Introduction. Soap bubbles naturally tend to minimize surface area for given volumes. This paper considers the two-dimensional analog of soap bubbles, seeking the way to fence in prescribed areas using the least amount of perimeter. For one prescribed quantity of area, the answer is of course a circle. This paper shows that for two prescribed quantities of area, the unique answer is the "standard double bubble" of Figure 1.0.1, and not the non-standard competitors admitted by the general existence theory ([**Al**], [**M**]) with disconnected bubbles or exterior (see Figure 1.0.2).



FIGURE 1.0.1. The standard double-bubble is the unique least-perimeter way to enclose and separate two prescribed areas.

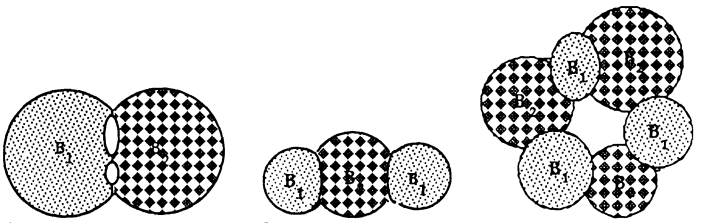


FIGURE 1.0.2. Some non-standard double-bubbles. (A) has connected bubbles but the exterior is disconnected. (B) has a connected exterior, but its B_1 bubble is disconnected. (C) has both disconnected bubbles and a disconnected exterior.

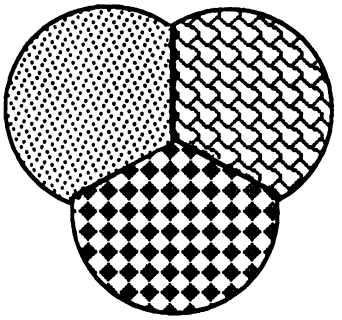


FIGURE 1.0.3. It is an open question whether the standard triple bubble is the least-perimeter way to enclose and separate three prescribed areas.

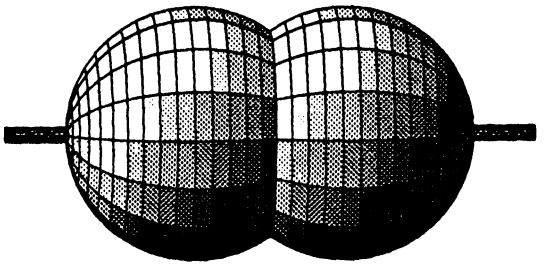


FIGURE 1.0.4. It is an open question whether the standard double bubble is the least-area way to enclose and separate two given volumes in $\mathbf{R}^3$.

For three or more areas, the question remains open (see Figure 1.0.3). In $\mathbf{R}^3$ it is also an open question whether the standard double bubble is the least-area way to enclose and separate two prescribed volumes (see Figure 1.0.4).

Proofs 1.1. The proofs first treat the case when there are no bounded components of the exterior trapped inside the configuration (Lemma 2.4). In this case, the bubble must be a combinatorial tree. In a non-standard double bubble, part of an extreme bubble can be reflected to contradict known regularity (cf. Figure 1.1.1).

The general case involves filling presumptive bounded components of the exterior to obtain a contradiction of certain monotonicity properties of the least-perimeter function.

Existence 1.2. F. J. Almgren [**Al**, Theorem VI.2] and [**M**] have established the existence of perimeter-minimizing bubble clusters in general

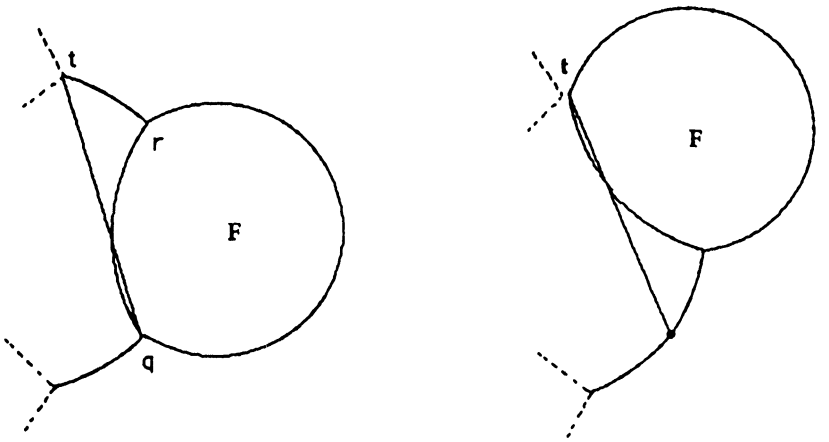


FIGURE 1.1.1. In the proof, presumptive extreme extra components can be reflected to contradict regularity. In the pictured reflection, there is a forbidden meeting of four arcs at the point t.

dimensions. Their results admit the possibility of *disconnected* bubbles and exterior. In the category of clusters with connected bubbles and connected exterior, existence remains open, although it is an easy mistake to think it obvious (cf. **[B]**).

The following regularity theorem, proved for $\mathbf{R}^3$ by Jean Taylor (**[Ta]**, **[ATa]**), appears in **[M]**.

REGULARITY THEOREM 1.3. *A perimeter-minimizing bubble cluster in* $\mathbf{R}^2$ *consists of arcs of circles (or line segments) meeting in threes at angles of* $120°$.

DEFINITIONS 1.4. A *cluster of bubbles* (or *bubble cluster*) in $\mathbf{R}^n$ is a collection of finitely many pairwise disjoint open sets, $B_1, B_2, B_3, \ldots, B_k$. Each open set $B_1, \cdots, B_k$ is called a *bubble.* We do not require bubbles to be connected.

Call a cluster of exactly two bubbles a *double bubble* and a cluster of exactly three bubbles a *triple bubble.*

In $\mathbf{R}^2$, a *standard double bubble* has three circular arcs (in this paper, a line segment is a circular arc) meeting at two vertices at angles of $120°$.

The *perimeter* of a cluster is given by the one-dimensional Hausdorff measure of the topological boundaries of the bubble:

$$\text{Haus}^1\left(\bigcup \partial B_i\right).$$

A cluster is *perimeter-minimizing* if no other cluster enclosing the same areas has less perimeter.

Acknowledgments. This paper consists of some main results of the Geometry Group [A] of the Williams College SMALL Undergraduate Research Project, summer 1990, a National Science Foundation site for Research Experiences for Undergraduates. Professor Frank Morgan advised the group. The work was further revised and developed in Joel Foisy's senior thesis [F] under advisor Dave Witte and by the SMALL 1991 Geometry Group, consisting of Thomas Colthurst, Chris Cox, Joel Foisy, Hugh Howards, Kathryn Kollett, Holly Lowy, and Steve Root. A very special thanks to Professors Morgan and Witte for all their help.

2. The double bubble in $\mathbf{R}^2$. Theorem 2.3 establishes the existence and uniqueness of the standard double bubble enclosing any two prescribed quantities of area. Lemma 2.4 shows that, if the exterior is connected, a perimeter-minimizing double bubble has connected bubbles. Proposition 2.8 shows that a perimeter-minimizing double bubble must have a connected exterior. Finally, the Main Theorem 2.9 establishes the standard double bubble as the unique perimeter-minimizing enclosure for two prescribed quantities of area.

We begin with some basic formulas and omit the easy computations.

PROPOSITION 2.1. *Let S be an edge of a perimeter-minimizing bubble cluster in $\mathbf{R}^2$. Define C to be the distance between the endpoints of S, with θ the angle between S and the line segment connecting its endpoints (see Figure 2.1.1). Then the radius of curvature R of S, the area A of the region between S and the line segment connecting its endpoints, and the length L of S are given by*

$$R(\theta, C) = \frac{C}{2\sin\theta}, \qquad A(\theta, C) = \frac{C^2(\theta - \sin(\theta)\cos(\theta))}{4\sin^2(\theta)}, \quad \text{and}$$

$$L(\theta, C) = \frac{C\theta}{\sin(\theta)}.$$

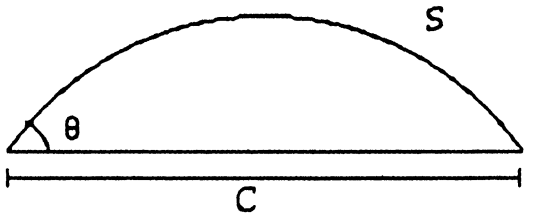


FIGURE 2.1.1. The circular arc S has radius of curvature R, area A and length L.

PROPOSITION 2.2. *If a perimeter-minimizing double bubble in* $\mathbf{R}^2$ *has connected bubbles and a connected exterior, then it is a standard double bubble.*

Proof. The whole cluster must be connected. Otherwise, sliding components together until their boundaries are tangent yields a contradiction of the Regularity Theorem 1.3.

In addition, Euler's formula concerning the vertices, edges, and faces of a planar graph gives $V - E + F = 1$. Since each vertex has degree 3, $2E = 3V$. We also know that $F = 2$. Solving these equations yields that $V = 2$ and $E = 3$. By the Regularity Theorem 1.3, the cluster indeed consists of three circular arcs all meeting in two points at angles of $120°$.

THEOREM 2.3. *For any two prescribed quantities of area, there exists a unique standard double bubble.*

REMARK. The proof shows that given a family of standard double bubbles with vertices a fixed distance apart, increasing the curvature on the separator arc decreases the area enclosed by the smaller bubble and increases the area enclosed by the larger bubble.

Proof. We will show that given any λ with $0 < \lambda \leq 1$, there exists, up to congruence, a unique standard double bubble B_1, B_2 such that the ratio $(\text{area}(B_1))/\text{area}(B_2))$ of the two areas it encloses is λ.

Consider a standard double bubble with the distance between its two vertices fixed to be 1. Note that its edges are circular arcs that meet at these two vertices. For the area underneath an arc of our enclosure, Proposition 2.1 yields

$$A(\theta) = A(\theta, 1) = \frac{\theta - \sin(\theta)\cos(\theta)}{4\sin^2(\theta)}.$$

A straightforward calculation shows that

$$A''(\theta) = \frac{2\theta(2 + \cos(2\theta)) - 3(\sin(2\theta))}{4\sin^4\theta}.$$

It can be routinely shown that $A''(\theta) > 0$ on $(0, \pi)$.

For any $\theta \in [0, \pi/3)$, an angle formed by a circular arc and the line segment of distance 1 that joins its endpoints, one can construct a standard double bubble (see Figure 2.3.1).

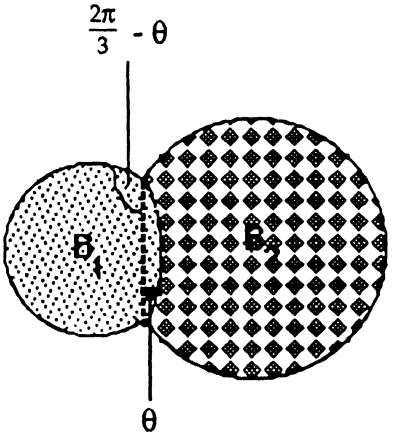


FIGURE 2.3.1. For any θ, one can construct a standard double bubble.

The enclosed areas satisfy

$$\begin{aligned} \text{area } B_1 &= A((2\pi/3) - \theta) + A(\theta), \\ \text{area } B_2 &= A((2\pi/3) + \theta) - A(\theta). \end{aligned} \tag{1}$$

For $\theta \in [0, \pi/3)$, any ratio of A_1 to A_2 will be uniquely represented. Indeed, let $F(\theta) = (\text{area}(B_1))/(\text{area}(B_2))$. Since $A''(\theta) > 0$ for θ in $(0, \pi)$, in $(0, \pi/3)$, area $B_1 = A((2\pi/3) - \theta) + A(\theta)$ is strictly decreasing and area $B_2 = A((2\pi/3) + \theta) - A(\theta)$ is strictly increasing. In general, increasing θ will decrease the area enclosed by the smaller

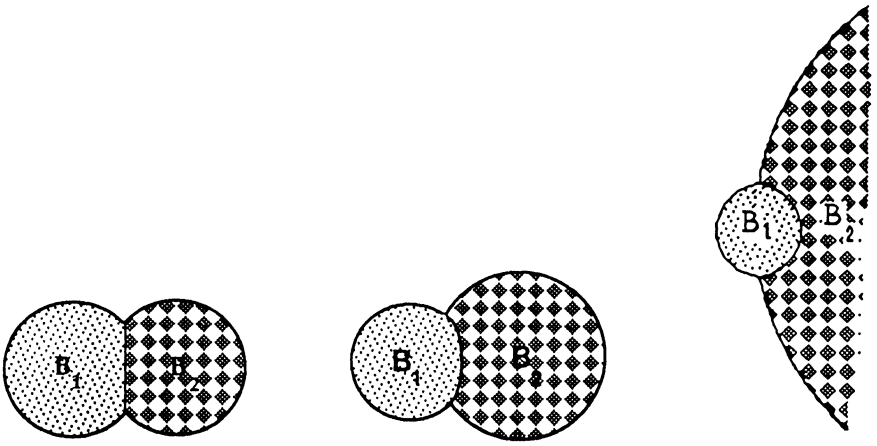


FIGURE 2.3.2. As θ varies, a unique standard double bubble represents every possible ratio of area(B_1) to area(B_2).

bubble and increase the area enclosed by the larger bubble. Thus $F(\theta)$ is strictly decreasing on the interval $[0, \pi/3)$. In addition, $F(0) = 1$ and $F(\theta) \to 0$ as $\theta \to \pi/3$. Thus $F: [0, \pi/3) \to (0, 1]$ is bijective (see Figure 2.3.2). Since F is bijective, a standard double bubble enclosing any two prescribed quantities of area uniquely exists for every value of $\theta \in [0, \pi/3)$.

We now have to show that any double bubble that contains bounded components of the exterior or disconnected bubbles is not perimeter-minimizing.

LEMMA 2.4. *A perimeter-minimizing double bubble whose exterior is connected must be standard.*

Proof. Let U be a perimeter-minimizing double bubble with connected exterior. If U is not standard, by Proposition 2.2, U has a disconnected bubble. We will show that U is not perimeter-minimizing.

Consider a graph formed by placing a vertex inside each bubble component of U, with an edge between vertices of adjacent components. For any U with a connected exterior, the corresponding graph has no cycles. Thus there will be a component of U that lies at an endpoint of the corresponding graph. It must have exactly two edges and exactly two vertices (see Figure 2.4.1).

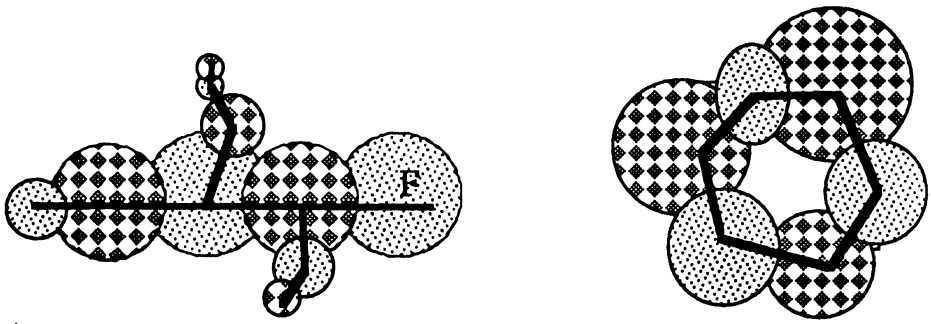


FIGURE 2.4.1. Since the exterior is assumed to be connected as in A, the associated graph has an endpoint in a component with two edges and two vertices. If the exterior were disconnected, then a cycle as in B could result.

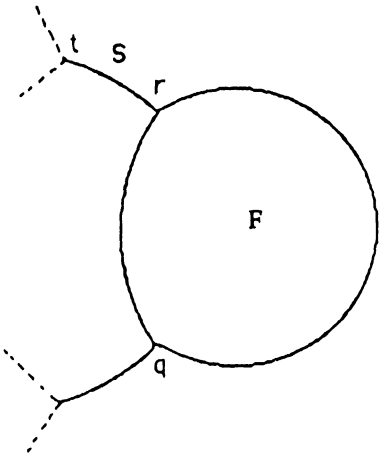


FIGURE 2.4.2. An extreme component F bounded by only two edges.

Let F be a component of U that has exactly two edges and exactly two vertices, r and q. Let t be a vertex of U that is adjacent to r but is not a vertex of F (see Figure 2.4.2). Let S be the edge connecting r and t.

Let $p = r$ and define a new bubble cluster, U_p, by replacing the component F by its reflection across the perpendicular bisector of the line segment qp $(= qr)$. If we let the point p move continuously along arc S from point r to point t and reflect component F and arc rp across the perpendicular bisector of line qp, the bubble cluster changes, but initially the perimeter and enclosed quantities of area remain constant.

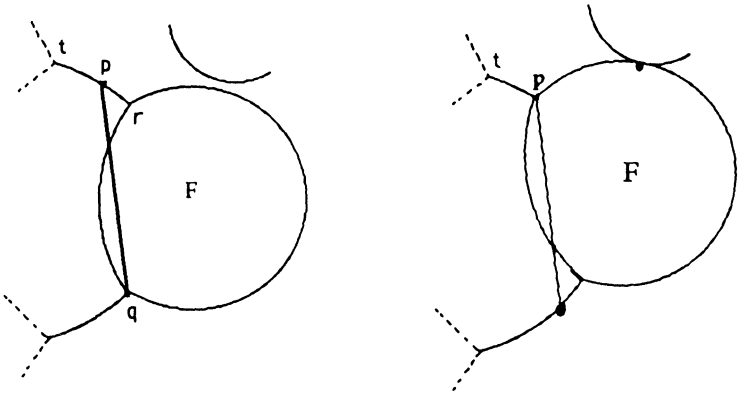


FIGURE 2.4.3. The reflected component F may touch another component, contradicting regularity.

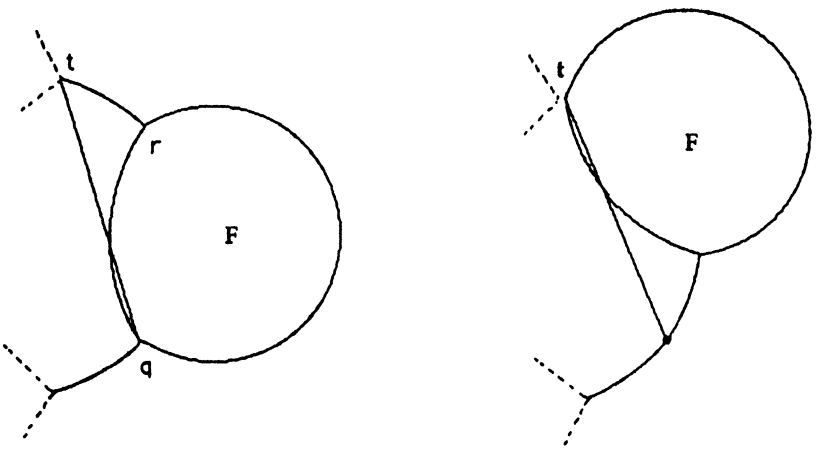


FIGURE 2.4.4. Otherwise the reflected component F eventually touches t, also contradicting regularity.

As p varies continuously from r to t, two things could happen: either there will be a point p for which the reflection will result in the touching of another bubble component and the creation of a new vertex with four edges leading to it (see Figure 2.4.3), or p will eventually coincide with point t, and there will be an instance of four edges meeting at a vertex (see Figure 2.4.4).

This operation creates a new bubble cluster of the same perimeter, enclosing the same prescribed quantities of area, that contradicts regularity. Therefore, the original bubble itself must not be perimeter-minimizing.

We will soon show that a perimeter-minimizing double bubble must have a connected exterior.

LEMMA 2.5. *Increasing the larger of the two prescribed areas enclosed by a standard double bubble will increase total perimeter.*

Proof. From Proposition 2.1, the length function for a circular arc with endpoints distance 1 apart and with θ, the angle between the segment connecting the endpoints and the arc, is

$$L(\theta) = \frac{\theta}{\sin(\theta)}.$$

The perimeter of the standard double bubble with angle θ between the line segment connecting its vertices (distance 1 apart) and the arc separating its two bubbles is given by

$$\text{perim}\,(\theta) = L(\theta) + L((2\pi/3) + \theta) + L((2\pi/3) - \theta).$$

A routine calculation shows that

$$L'(\theta) = \frac{\sin\theta - \theta\cos\theta}{\sin^2\theta} \quad \text{and}$$
$$L''(\theta) = \frac{\theta\sin^2\theta + 2\theta\cos^2\theta - \sin 2\theta}{\sin^3\theta}. \tag{1}$$

It can easily be shown that $L'(\theta) > 0$ on $(0, \pi)$, and thus $L(\theta)$ is increasing on $[0, \pi/3)$. In addition, $L''(\theta) > 0$ on $(0, \pi)$; thus on $[0, \pi/3)$, $L((2\pi/3) + \theta) + L((2\pi/3) - \theta)$ is increasing. This implies that $\text{perim}(\theta)$ is increasing on $[0, \pi/3)$.

Thus increasing θ will increase the total perimeter. In addition, it follows from Theorem 2.3 (cf. Remark) that increasing θ will decrease the area enclosed by the smaller bubble and increase the area enclosed by the larger bubble. By scaling up the double bubble until the smaller bubble contains its original area, only the area enclosed by the larger bubble will increase. In the process, total perimeter also increases.

DEFINITION 2.6. For two prescribed quantities of area, A_1 and A_2, let $P(A_1, A_2)$ be the perimeter of the least-perimeter double bubble enclosing areas A_1 and A_2. Let $P_0(A_1, A_2)$ be the perimeter of the standard double bubble enclosing areas of size A_1 and A_2.

LEMMA 2.7. *For any fixed* $A_1, A_2 \geq 0$, *the function* $P(A, A_2)$ *has a minimum for* $A \in [A_1, \infty)$.

Proof. By the isoperimetric inequality, $P(A, A_2) \geq \text{perim}(D)$, where D is a disk of area A. Hence $P(A, A_2) \to \infty$ as $A \to \infty$. Since P is continuous, P has a minimum for $A \in [A_1, \infty)$.

PROPOSITION 2.8. *The exterior of a perimeter-minimizing double bubble must be connected.*

Proof. Given two quantities of area, A_1 and A_2, without loss of generality assume $A_1 \geq A_2$. Suppose that the exterior of a perimeter-minimizing double bubble B_1, B_2 enclosing A_1 and A_2 is disconnected. By Lemma 2.7, we can choose some $A_1' \in [A_1, \infty)$ that minimizes $P(A_1', A_2)$. In particular, $P(A_1', A_2) \leq P(A_1, A_2)$.

We now show that if B_1', B_2 is the perimeter-minimizing enclosure of the quantities of area A_1' and A_2, respectively, then the exterior of B_1', B_2 is connected. Suppose, to obtain a contradiction, that the exterior is disconnected. By incorporating the bounded components

of the exterior into the bubble B_1' and then removing the edges that were separating the bounded components of the exterior from B_1', a new double bubble, B_1'', B_2, with less perimeter than B_1', B_2 is formed. This contradicts the choice of A_1'.

Thus $B_1' \neq B_1$ and hence $A_1 < A_1'$. By Lemma 2.5, $P_0(A_1, A_2) < P_0(A_1', A_2)$. By Lemma 2.4, $P_0(A_1', A_2) = P(A_1', A_2)$. By definition, $P(A_1, A_2) \leq P_0(A_1, A_2)$. In summary:

$$P(A_1, A_2) \leq P_0(A_1, A_2) < P_0(A_1', A_2) = P(A_1', A_2) \leq P(A_1, A_2).$$

This is a contradiction. We conclude that the exterior must be connected.

MAIN THEOREM 2.9. *For any two prescribed quantities of area, the standard double bubble is the unique perimeter-minimizing enclosure of the prescribed quantities of area.* (*See Figure* 2.9.1.)

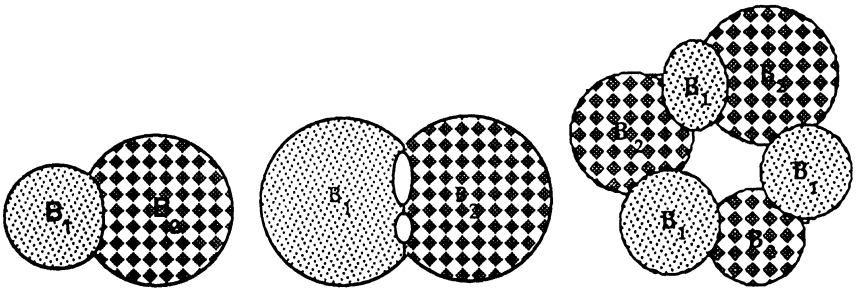


FIGURE 2.9.1. Theorem 2.9 shows that a perimeter-minimizing double bubble must look like the first one, not the second two, which have disconnected bubbles or exteriors.

Proof. By Theorem 1.6, we know that a perimeter-minimizing double bubble exists. By Proposition 2.8, the exterior of this double bubble must be connected. Then, by Lemma 2.4, the bubble cluster must be standard. Therefore, only a standard double bubble is perimeter-minimizing. By Theorem 2.3, it exists uniquely for any two prescribed quantities of area.

Lemma 2.5 showed that increasing the larger quantity of area enclosed by a standard double bubble increases perimeter. It follows from our main theorem that increasing either quantity of area enclosed by a standard double bubble increases perimeter.

COROLLARY 2.10. *Increasing either given area A_1, A_2 increases the perimeter of the perimeter-minimizing double bubble.*

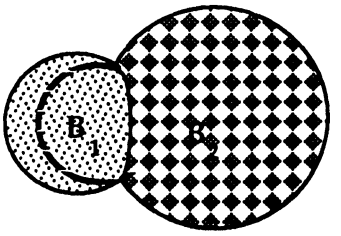


FIGURE 2.10.1. After we shorten an outside arc, the resulting double bubble has less perimeter, and B_1 has less area.

Proof. If not, for some A_1, A_2 some slight increase in, say, A_1 decreases the least perimeter of the minimizing double bubble. Now A_1 can be decreased back to its original value by shortening an outside arc, as in Figure 2.10.1, and further decreasing perimeter, contradicting the minimizing property of the original double bubble.

Added in proof. There has been recent progress on the existence of connected bubbles [M] and on the triple bubble:

Chris Cox, Lisa Harrison, Michael Hutchings, Susan Kim, Janette Light, Andrew Mauer, Meg Tilton, *The shortest enclosure of three connected areas in* $\mathbf{R}^2$, NSF SMALL undergraduate research Geometry Group, Williams College, 1992.

Also see the survey:

Frank Morgan, *Mathematicians, including undergraduates, look at soap bubbles*, Amer. Math. Monthly, (1993), in press.

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