Fermat's method of "descente infinie" - Proof of Fermat's Last Theorem for n=4

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Introduction

Fermat proved the impossibility of finding positive integer solutions of the equation $x^4 + y^4 = z^4$ by a method which he claimed would 'enable extraordinary developments to be made in the theory of numbers' [1], His method was based upon the idea of using a supposed solution to find another solution, smaller than the first. Proceeding in this fashion he could therefore obtain an infinite series of smaller and smaller solutions. At this point he concluded his proof as follows:

'This is, however, impossible because there cannot be an infinite series of numbers [positive integers] smaller than any given [positive] integer we please. The margin is too small to enable me to give the proof completely and with all details [1].'

The margin referred to here was in Fermat's copy of the Arithmetica, an ancient Greek text written by Diophantus of Alexandria. The problems given in this text provided the stimulus for many of Fermat's discoveries in number theory. The equation $x^4 + y^4 = z^4$ arose from problems concerning right-angled triangles

Fermat's proof

Fermat gave sufficient details of his proof to enable it to be reconstructed. However, the purpose of this note is not to reproduce the original proof faithfully but to modify and simplify it whilst retaining the historical connection with right-angled triangles.

Theorem: A right-angled triangle with rational sides cannot have two sides each with a length equal to a square or twice a square.

Proof. Suppose such a triangle did exist. By scaling by a square number we can further assume the sides to have integer lengths. Then, any prime factor p of two of the lengths would be a factor of all three lengths. Furthermore, if p were odd, then p^2 would be a factor of two of the lengths and thus of all the lengths. By cancelling down as necessary, we can therefore assume the lengths to be coprime in pairs. Then, by Euclid's formula, there are coprime positive integers a and b, one odd and one even, such that the triangle is as shown.

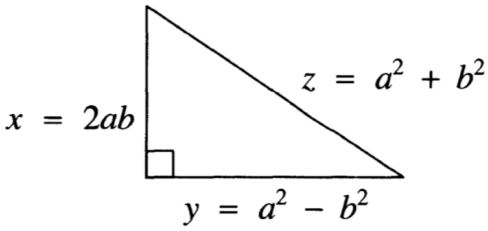


FIG. 1:

Now y and z are both odd and so neither can be twice a square. If they were both squares then the triangle shown below would be a new solution.

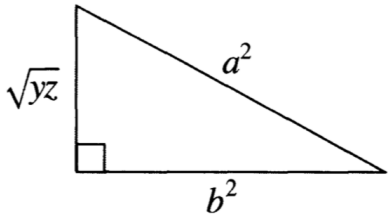


FIG. 2:

Otherwise, just one of y and z is a square and $x = 2ab$ is a square or twice a square. Then each of a and b is a square or twice a square and one of the triangles shown below would be a new solution.

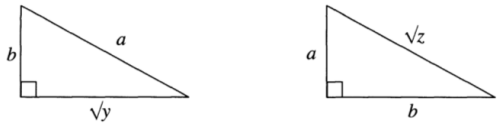


FIG. 3:

In all cases, the new solution has a smaller hypotenuse than the original triangle. Since there cannot be an infinite series of decreasing hypotenuses, there can be no triangle with the required properties.

Concluding Remarks

A positive rational solution of any of the equations

$$\begin{aligned} x^4 + y^4 &= z^4 \\ x^4 - y^4 &= z^4 \\ x^4 + 4y^4 &= z^4 \\ x^4 - 4y^4 &= z^4 \end{aligned} \tag{1}$$

would give rise to a right-angled triangle contradicting the Theorem. All of these equations are therefore insoluble.

It is worth noting that it is not obvious how to use a supposed solution of $x^4 + y^4 = z^4$ to find another solution of the same equation. However, by strengthening the conjecture as shown above it was possible to carry out the proof relatively simply. This illustrates an important feature of Fermat's method of proof; that the choice of conjecture can be crucial to the success of the method. Sometimes it can be much easier to prove a stronger result than the one which is needed.

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[1] Sir Thomas L. Heath, Diophantus of Alexandria, Dover, 1964, p. 293.