the SHELDON CONJECTURE



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The Big Bang Theory is a CBS sitcom created in 2007. It follows the escapades of five friends: Sheldon Cooper, physicist; Leonard Hofstadter, physicist; Howard Wolowitz, aerospace engineer; Raj Koothrappali, astrophysicist; and Penny, a waitress and aspiring actress who lives across the hall. In the 73rd episode of the show, we hear the following conversation [1].

Sheldon: What is the best number? By the way, there's only one correct answer.

Raj: Five million, three hundred eighteen thousand, eight?

Sheldon: Wrong. The best number is 73. You're probably wondering why.

Leonard: No.

Howard: Uh-uh.

Raj: We're good.

Sheldon: Seventy-three is the 21st prime number. Its mirror, 37, is the 12th, and its mirror, 21, is the product of multiplying, hang on to your hats, seven and three. Eh? Eh? Did I lie?

Leonard: We get it. Seventy-three is the Chuck Norris of numbers.

Sheldon: Chuck Norris wishes. In binary, 73 is a palindrome, 1,001,001 which backwards is 1,001,001, exactly the same. All Chuck Norris backwards gets you is Sirron Kcuhc.

Sheldon is correct in each of his statements regarding 73. We will call a number that satisfies these properties (which we define explicitly below) a *Sheldon prime.* Implicit in Sheldon's assertion that 73 is the best number is that 73 is the only Sheldon prime. We call this the *Sheldon conjecture.*

Although we cannot prove the conjecture, we show that a counterexample, if one exists, must be very large.

The Mirror Property

We begin by defining some notation that will help us simplify this problem. First, let $p(n)$ be the nth prime number. For example, $p(1) = 2$, $p(8) = 19$, and $p(582{,}161) = 8{,}675{,}309$.

The *mirror* of a natural number x, which we denote $m(x)$, is the number obtained by reversing the order of the digits of x. So $m(8{,}675{,}309) = 9{,}035{,}768$ and $m(1{,}000) = 1$.

Some numbers satisfy the very special property that $x = m(x)$. These numbers are their own mirrors and are called *palindromic numbers. Palindromic primes*, such as 101 or 373, are rare and have been studied in their own right [5]. There are infinitely many palindromic numbers, but it is an open question as to whether there are infinitely many palindromic primes. As of November 2014, the largest known palindromic prime was 474,501 digits long [6].

With these definitions in hand, we are ready to state the first property of a Sheldon prime.

Definition. The nth prime number r satisfies the *mirror property* if $m(r)$ is the $m(n)$th prime number. That is, $m(p(n))=p(m(n))$.

There are actually two statements in the above property. The first is that $m(r)$ is a prime number (which is not guaranteed). The second is that $m(r)$ is the $m(n)$th prime number (definitely not guaranteed). We will look at each part of this property.

Palindromic primes satisfy the first part of the property, but they are not the only ones. For example, 389 and $m(389) = 983$ are both prime. Although there are several obvious conditions that would preclude a prime from satisfying this property, such as starting with a 2, 4, 5, 6, or 8, primes that satisfy this first half are common. Of the first 10 million primes, 14.99



Photos courtesy CBS

From left: Leonard, Sheldon, Raj, and Howard. Sheldon implies that 73 is the only number with the properties he lists.

percent of them have prime mirrors.

The second half of the mirror property states that $m(r)$ is the $m(n)$th prime number. This requirement is much stronger than the first. Primes such as 2, 3, 5, 7, and 11 satisfy this property because both n and $p(n)$ are palindromes. Also included in this list is 73, the 21st prime, and 37, the 12th. Looking at the first 10 million primes again, we find only one prime greater than 73 that satisfies the mirror property: $p(8{,}114{,}118) = 143{,}787{,}341$. In this example, as was the case with the "trivially" palindromic one-digit primes and 11, both n and $p(n)$ are palindromes.

The Product Property

For any natural number x, let $\Pi(x)$ denote the product of the digits of x. For instance, $\Pi(73) = 21$, $\Pi(419) = 36$, and $\Pi(8{,}675{,}309) = 0$. We are now ready to state the second property of a Sheldon prime.

Definition. The nth prime number r satisfies the *product property* if $\Pi(r) = n$. That is, $\Pi(p(n)) = n$.

First, notice that if the prime r contains a zero as one of its digits, then it will not satisfy the product property because $\Pi(r) = 0$. This fact significantly restricts the number of primes that can satisfy the product property. Suppose q is a k-digit prime. Clearly q cannot begin with a 0 and it cannot end with a 0 (or else it is not prime). Assuming that for the remaining $k-2$ digits the numbers 0–9 are equally likely, the probability that q does not contain a 0 is $(9/10)^{k-2}$. There is a relatively high probability that a small prime will not contain a 0 digit. For larger primes, the probability is nearly zero. For one that is 512 digits long, the probability is approximately 4.6×10^{-24}.

The quantity $\Pi(x)$ is the product of the digits of x. Because each digit of x is one of $\{0,1,2,3,4,5,6,7,8,9\}$, 2, 3, 5, and 7 are the only possible prime factors of $\Pi(x)$. In number theory, integers that have a small number of factors are known as *smooth numbers*. Formally, integers whose prime factors are all less than or equal to y are called *y-smooth* [4]. For instance, 72 is a 3-smooth number because $72 = 2^3 \cdot 3^2$, and 125 is 5-smooth but not 3-smooth.

So $\Pi(x)$ is a 7-smooth number for all x. In particular, if n has a prime factor greater than 7, then $p(n)$

cannot satisfy the product property. This greatly reduces the number of integers in the running to be Sheldon primes.

We know that the 21st prime, 73, satisfies the product property, and so does the 7th prime, 17. The only other instance in the first 10 million primes is the 181,440th prime, 2,475,989 (because $2 \cdot 4 \cdot 7 \cdot 5 \cdot 9 \cdot 8 \cdot 9 = 181,440$).

Sheldon Primes

We may now define a Sheldon prime.

Definition. A natural number that satisfies the mirror property and the product property is called a *Sheldon prime.*

Suppose $p(n)$ is a Sheldon prime. The product property implies that $n = 2^i 3^j 5^k 7^l$ for some $i, j, k, l \geq 0$. But the mirror property enables us say more about the prime factorization of n.

For most natural numbers, n, n and $m(n)$ have the same number of digits. But each 2-5 pair in the prime factorization of n yields a zero at the end of n and decreases the number of digits of $m(n)$ by one. We claim that if 100 divides n, then $m(n)$ is too small for $p(n)$ to satisfy the mirror property. To justify this claim, we need to know more about the relative sizes of n and $p(n)$.

The famous prime number theorem, proven by Jacques Hadamard around 1896, states that there are approximately $x / \ln x$ prime numbers less than x. A corollary of the prime number theorem states that $p(n)$ is approximately $n \ln(n)$. For instance, this formula states that the millionth prime is approximately $1,000,000 \cdot \ln(1,000,000) = 13,815,510$. In actuality, $p(1,000,000) = 15,485,863$.

This formula gives good approximations, but not good enough for our needs. We need a more precise statement about the size of $p(n)$. In [2] and [3], we are provided the following upper and lower bounds of $p(n)$. For $n \geq 6$,

$$n \ln\left(n / e\right) + n \ln(\ln(n)) < p(n) < n \ln(n) + n \ln(\ln(n)).$$

For our purposes, we care less about the value of $p(n)$ than about the number of digits of $p(n)$. To compute the number of digits of a number, we need a basic result involving logarithms. In particular, $\lfloor \log_{10}(x) \rfloor + 1$ is the number of digits of a natural number x. Recall that $\lfloor \cdot \rfloor$ is the *floor function*; $\lfloor x \rfloor$ is the greatest integer less than or equal to x. For instance, $\lfloor \log_{10}(294) \rfloor + 1 = \lfloor 2.468 \rfloor + 1 = 3$ is the number of digits of 294.



Photos courtesy CBS

Sheldon's comments about Chuck Norris are icing on the cake.

Now suppose 100 divides a natural number n. Then $m(n)$ has at least two fewer digits than n. On the other hand, $p(n)$ and $m(p(n))$ have the same number of digits because $p(n)$ is a prime, which cannot end in a 0.

To show that the number of digits of $p(m(n))$ is too small to correspond to the number of digits of $m(p(n))$, we will use the upper and lower bounds given above. Using the upper-bound formula for $p(m(n))$ and the fact that $m(n) < n / 10$, we have

$$\begin{aligned} p(m(n)) &< m(n) \ln(m(n)) + m(n) \ln(\ln(m(n))) \\ &< (n / 10) \ln\left(n / 10\right) + (n / 10) \ln\left(\ln\left(n / 10\right)\right) \\ &< (n / 10) \ln\left(n / 10\right) + (n / 10) \ln\left(\ln\left(n\right)\right). \end{aligned}$$

Using the properties of the logarithm and the floor function, we find the number of digits of $p(m(n))$ is *at most*

$$\begin{aligned} &\left\lfloor \log_{10}\left((n / 10) \ln\left(n / 10\right) + (n / 10) \ln\left(\ln\left(n\right)\right)\right)\right\rfloor + 1 \\ &= \left\lfloor \log_{10}\left(n \ln\left(n / 10\right) + n \ln\left(\ln\left(n\right)\right)\right)\right\rfloor. \end{aligned}$$

Using the lower-bound formula for $p(n)$, we have

$$\begin{aligned} p(n) &> n \ln\left(n / e\right) + n \ln(\ln(n)) \\ &> n \ln\left(n / 10\right) + n \ln(\ln(n)). \end{aligned}$$

Thus, the number of digits of $p(n)$ is *at least*

$$\left\lfloor \log_{10}\left(n\ln\left(n/10\right)+n\ln(\ln(n))\right)\right\rfloor+1.$$

So, the number of digits of $p(n)$, and hence $m(p(n))$ because they have the same number of digits, is at least one more than the number of digits of $p(m(n))$. Thus $p(m(n)) \neq m(p(n))$.

Consequently, for $p(n)$ to be a Sheldon prime, n must be a 7-smooth number not divisible by 100. Only 3,039

Is there a Sheldon prime other than 73?

of the first billion numbers have this form. In other words, this criterion alone rules out 99.9997 percent of the first billion primes as possible Sheldon primes.

Sheldon-Norris Primes

The astute reader will notice that we have not addressed the Chuck Norris property of 73—that it is a palindrome in binary. After much discussion, the authors decided that Sheldon mentions this property solely as icing on the cake, rather than a pillar of his mathematical argument.

However, a quick check of the first 10 million primes shows, not surprisingly, this is yet another restrictive property, with only 0.0019 percent of those primes being binary palindromic. If the chance of finding another Sheldon prime is small, the chance of finding another Sheldon-Norris prime is even smaller.

So where does all this analysis lead us? Sheldon's claim focused on two properties of the number 73. One of them, that 73 is the 21st prime and its mirror 37 is the 12th, is very restrictive. However, we found an example of a large prime that satisfies the mirror property, $p(8{,}114{,}118) = 143{,}787{,}341$.

The second property, the product property, was that $\Pi(73) = 7 \cdot 3 = 21$. Again, this property is incredibly restrictive, requiring the n under consideration to be a 7-smooth integer not divisible by 100. In addition, the prime cannot contain any zeroes as digits. However, our nontrivial example of $p(181,440) = 2,475,989$ shows that the product property can be satisfied by larger numbers.

Is there a Sheldon prime other than 73? These authors believe it unlikely. Can you prove the conjecture? Or can you find a counterexample? ■

Further Reading

[1] "The Alien Parasite Hypothesis," *The Big Bang Theory,* season 4, episode 10, *http://bit.ly/1HYFGkT.*

[2] E. Bach and J. Shallit, *Algorithmic Number Theory 1*, MIT Press (1996) 233.

[3] P. Dusart, The kth prime is greater than $k(\ln(k)+\ln(\ln(k))-1)$ for $k \geq 2$, *Mathematics of Computation* **68** no. 225 (1999) 411–415.

[4] A. Granville, Smooth numbers: computational number theory and beyond, *Algorithmic Number Theory* **44** (2008) 267–316.

[5] T. P. Hill, The significant-digit phenomenon, *Amer. Math. Monthly* **102** no. 4 (1995) 322–327.

[6] P. Ribenboim, *The Little Book of Bigger Primes*, Springer (2004).

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