Hierarchical Introspective Logics

This research was initially stimulated by the announcement in June 1993 of a proof of the FLT conjecture ("Fermat's Last Theorem"). Once again a mathematical question that might conceivably have been unanswerable was being answered by the efforts of humans to find an acceptable proof of a standing conjecture.

Earlier the famous 4 Color Conjecture had yielded to an attack aided by computers.

So the question that arose in my mind was that of whether or not there actually exists any mathematically interesting true mathematical assertion (such as, for example, perhaps the Riemann Hypothesis) which cannot be proved and which must ultimately be accepted as a new axiom if mathematics is to proceed in terms of assertions and proofs (Satz; Beweis).

I was of course aware of Goedel's establishment of the existence of assertions concerning the natural integers that are not provable modulo the proof rules of specific given formal systems and also of the general results on undecidability which also imply the existence of such assertions that are true but not provable in the context of a given formal system.

The specific sort of formula that is produced by the construction of Goedel, or by Rosser's variation on Goedel's construction, is true but not provable within the formal system on the basis of which it was constructed. However an acceptable mathematical proof, in verbal form, was given by Goedel of the truth of his constructed formula provided that it could be assumed that the formal system at issue was ideally consistent.

Thus the Goedel formula or assertion is not

really of itself mysterious; it is provably true, on a common-sense basis, and it shows therefore that the formal system studied must have been inadequate to achieve the ideal goal of completeness. (It can be remarked that logical systems of a more limited application, specifically systems not adequate for the proof of general propositions of "number theory", for example the "calculus of sentences", can be complete. For systems with a narrower scope of applicability completeness does not introduce paradoxes.)

If a Goedel or Goedel-Rosser assertion is added to one of these incomplete systems as an axiom then one obtains a new logical system with similar characteristics to the original. It is thus again incomplete and again a Goedel assertion can be found within it, a formula that is true but that is not provable by the formal machinery of the system.

But better than adding the Goedel or the Goedel-Rosser assertion to the initial system as an axiom one can instead add to it an axiom of consistency. That is one can add an axiom stating that the initial system was formally consistent. This does not also say that the new system including the added axiom is itself consistent.

The axiom asserting formal consistency could be expressed with the use of a scheme of Goedel numbering for the formulae expressible in the initial system and also for proofs possible in that system. Then it simply asserts that no such proof can result in a formal proof of falsehood (which is represented by the second truth symbol of the pair of T and F).

It is known that adding such an axiom of formal consistency to the initial system makes it then possible, using the added axiom, to prove either the Goedel assertion or the Rosser assertion for that system (the initial system).

Extended Systems

This sort of supplementary axiom, which serves naturally to deal with the incompleteness of a logical system, this supplementation produces a natural extension of the original system. This type of extension process was studied in the mid- and later 30's by Turing in a project that became his Ph.D. thesis at Princeton.

One thing that is notable is that when an axiom is added that asserts formal consistency of an initial system that this added axiom is not itself included in relation to the consistency issue; so the extended system formed by the addition of that axiom is not itself asserted to be consistent by the added axiom. And thus, for the newly formed extended system to be itself axiomatically stated to be consistent requires the introduction of a second added axiom in a parallel proceeding.

Turing considered this process in his paper (the thesis) "Systems of logic based on ordinals" published in 1939 in Proc. Lond. Math. Soc. But Turing did not construct his systems, his towers of extension, on a basis of "ideal ordinals", corresponding to the mathematician's traditional way of thinking of mathematical objects, but rather he worked in terms of arrays of ordinals which were indexed somehow by ordinary integers and described by a table of ordering by which the order relation of any two of the indexed entities was specified. And this order table was required to be specified by recursive functions.

When I began to think about the same general theme, that of axiomatically extending an initially specified logical system so as to "correct" or fill in for the incompleteness of the type discovered by Goedel, it occurred to me that Turing's concept of how ordinals could be used may have been intrinsically too limiting and that it would be

desirable to explore what might be done with a less restrictive concept of ordinal numbers than that of the recursively presented sets of them used by Turing.

There are many logical "paradoxes" that can arise if we speak loosely about ordinals and their definability. On the other hand I do think that Turing's concept of usable ordinals was too restrictive to allow the related extension scheme to proceed to really interesting results relating to incompleteness.

Another Form of Incompleteness

The discovery of incompleteness by Goedel was supplemented in the 30's by the work of Turing, Church, and others who showed that there was "undecidability" with regard to the totality of assertions possible within a formal system of the same general category as those for which Goedel's discovery applied. This showed that it was not possible, systematically, to decide which assertions were true and which false. And consequently to that there could not exist proofs for all of the true assertions since that would make possible the systematic decision process. Thus a true but unprovable assertion MUST EXIST although this route, via "undecidability", did not produce any specific example.

But much more recently a paper was published by Paris and Harrington showing that a popular form of Peano arithmetic called PA- (upper suffix -) was incomplete because of the lack of a sufficiently strong "axiom of infinity". This result was very interesting since it showed that incompleteness could have this sort of a connection with "knowledge of infinity" and with the possibilities for infinite arguments such as proofs by induction.

Of course there are many long known paradoxes,

or potential logical traps, that arise from talking about infinity. One of these can be illustrated by imagining two mathematical logicians named Adams and Bentley. So Bentley says "Let beta be the limit ordinal (number) which is the least ordinal that is larger than any ordinal alpha definable in the formal language of the system of Adams. This illustrates the paradoxes that inherently arise because we seek to "talk about" infinite concepts but we have only finite resources to use for the actual purpose of communication in words. Any book written by humans has a text that contains only a finite amount of information, a certain number of "bytes" as it were. And this illustrative paradox relates to the classical "Burali-Forti paradox" of logic and set theory.

Dangers of Paradoxes, Personal Caution

I am speaking about a research project that is not fully complete since I have not yet written up and submitted for publication any paper or papers describing the work. Also the details of what axioms to use and how to select the basic set theory underlying the hierarchical extension to be constructed are not fully crystallized.

I have also a great fear of possible error in studying topics in this area. It is not rare, historically, for systems to be proposed that are either inconsistent or that have unexpected weaknesses. So I feel that I must be cautious and proceed without rushing to a goal. And this psychology of fear has also inhibited me from consulting other persons expert in logic before I could feel that I had gotten my own ideas into good shape.

The Overview Concept

It is notable that Goedel's construction of an unprovable, yet true, proposition in a generic sort of formal system is accompanied also by a proof, under reasonable hypotheses, of the truth of that "unprovable" proposition. How is this achieved? Well, of course, the proof does NOT occur WITHIN the formal system for which the Goedel proposition was constructed, rather it is given verbally in the normal style of argument for published mathematical papers.

But it can be observed that if another logical system were constructed with the specific goal of studying the first system and the proofs possible within it and if this "overview" system also included axioms to the effect that the machinery of procedure of the first system is truth-preserving and that the axioms of the first system are true then it would become provable, in this overview system, that the first system is consistent and also "Omega-consistent". And from these proofs would follow, in the overview system, the proof of the truth of a Goedel assertion for the first system.

These observations relate to what Turing studied in his paper "Systems of logic based on ordinals". In hindsight one could say that Turing's concept of extension was (italicize) THE UNIQUE AND NATURAL CONCEPT so far as the range of the finite ordinal numbers is concerned. This results in an ascending ladder of systems in which each successor is more complete than any of its predecessors but this ladder ascends only through the finite ordinal numbers, 0th, 1st, 2nd, 3rd, 4th, 5th, ... nth, ... and does not reach any transfinite levels.

Of course Turing did actually concern himself with transfinite ordinal levels but he did not have a way of uniquely describing them. (This was the

"invariance" problem in the terminology of his paper.)

Although there seems to be essentially no confusion about the naming or proper description of finite ordinal levels, beyond that there are serious difficulties that are classically known. Thus mathematically there should be a non-enumerable totality of ordinal numbers each of which has only an enumerable set of predecessor ordinals. Then how many can we expect to properly name or describe in the words of any precise mathematical language? And what can be said about the description: "the least ordinal not thus properly describable"? Relating to this is the mathematical property of the ordinals that any subset of them has a least member. This shows by illustration how difficult it is to form an ideal concept of definable ordinals or of canonical names for ordinals.

Turing's hierarchy of levels in any one of his systems was based on association of the levels with ordinals which were themselves associated with natural integers through a recursive description of a subset of all ordinals. Unfortunately this specification did not have uniqueness, or "invariance" as he called it, a property that he knew was most desirable.

In my own thinking, after a long period of study and after encountering the problem that although one could talk about "ideal" (or mathematical) ordinals one would need names for them in order to be able to refer to them in formulae of a formal system. Ultimately I came to the idea that instead of associating the levels of a system with ordinal numbers they could instead be associated with definitions of ordinals. This is not the same as indexing the levels by ordinals. Two quite different definitions might easily define the same mathematical ordinal yet also it could be very

difficult to determine the precise comparative relationship of two different definitions, each of them stating the definition of a specific mathematical ordinal.

And it is also easily observed that any specified language would enable the formation of only an enumerable array of finite formulae to serve as definitions for ordinals while mathematically the number or cardinality of the ordinals is unlimited. (And as was mentioned above, even the set of ordinals where each has only enumerably many predecessors, this set should itself be of higher cardinality and non-enumerable.)

An Hierarchy of Levels

So we evolved the idea that the formal system serving to extend a given basic system (or "ground level") could be structured by making use of special functions enabling references to levels. In effect, if we used references indicating a higher level
then it would become permissible to "overview" the formulae and procedures of a lower level.

And the basic idea, at least at the beginning of the construction of an hierarchy of extension levels, is quite like Turing's concept in that the first extension level, the first level above the ground level, incorporates the possibility to prove as a theorem the Goedel assertion for the logical system of the "ground level".

The first higher level, corresponding in our approach to any acceptable definition of the first ordinal number, would be such that the Goedel assertion from the ground level would now become provable if that Goedel assertion had been simply added as an axiom usable on the higher level. Or instead it would be effective to introduce as an

axiom an assertion of formal consistency of the ground level logic. This type of assertion can be seen to imply the assertion of Goedel's type. Formal consistency is the assertion to the effect that
"on the ground level nothing false can be proved."

The verbal argument for the truth of Goedel's formula depends in effect on an overview of the specific formal system in which it is stated and on the basis of the axioms, etc. of which it was constructed. Also this argument depends on the assumption that that formal system is actually perfectly consistent. Our idea of extension and of "hierarchical introspective logics" is that there is to be considered an hierarchy of levels of logic such that higher levels have an "overview" of the proceedings and results obtainable on lower levels. Thus the totality is "introspective" because it is looking inward to study itself but this self-study is not unrestricted. And effectively a higher level is supposed to be able to see the truth of a Goedel assertion deriving from a lower level by a process analogous to the verbal argument originally known for the truth of the Goedel assertion in a formal system although there is no proof of that assertion in that original formal system.

A logical system cannot effectively state its own consistency; this relates to the incompleteness originally found by Goedel. But one logical system CAN easily state the formal consistency of another system. We use this idea in creating our hierarchy of levels so that on a higher level it is possible to, in effect, assume the validity and the consistency of proceedings possible on lower levels.

Axioms and Special Functions

(This portion of the talk should make use of

transparencies presenting the axioms or axiom schemata and the special functions introduced.)

Our design for the extension hierarchy has the feature that certain special logical functions are introduced as components of formulae on all levels of the hierarchy except the ground level, the ground level being simply the formal logic that we are extending. The initial or ground level system should be adequate to be of the sort studied by Goedel and to be able to deal with "number theory", "functional calculus", and "set theory". And we prefer the viewpoint of Zermelo, that sets can be based on ur-elements which are not themselves sets. Also we like to think of ordinal numbers as being basically a special type of mathematical objects such that they satisfy appropriate axiomatic properties. Thus the ordinals, as mathematical entities, are analogous to the quaternions. Of course this doesn't really matter, it's a question of taste.

We assume, for convenience, that a specific "canonical" "Goedel numbering" has been chosen, both for formulae (wff's) and for "strings" of formulae that are presented as proofs. This is with regard to the extended system, not just the "ground level". Then each wff or string of wff's has its index which is simply one of the natural integers. And the special functions that we introduce are to be understood to use these index integers as arguments. We favor the convention that the formula or string that is indexed may be substituted for the indexing integer in an instance of any of these special functions.

(Give oral verbal description with display of transparencies for axioms and special functions.

Explain that ORDDEF("delta") is to be affirmed or true ONLY when "delta" is a definition in a standard form using only notation/symbols appropriate for the ground level. And explain also that such a definition, to be approved, must be recognizable as PROVABLY the def. of an unique ordinal by the logic of the GROUND LEVEL. And remark that the function ORDDEF(~~) is NOT recursive but is definable from a set of recursive functions relating to provability and form.)

Regarding axioms, we are not sure that we have the final set of main axioms and also we see some overlapping or redundancy of axioms dealing with the KT(~~,~~) or "known true" function. It is also a technical need that certain axioms should be present that simply assert that the special functions are what they are supposed to be.

Inductive and General Incompleteness

The incompleteness found and exhibited in the paper of Paris and Harrington has a relation to proofs by induction and the possibility of achieving desired results by that means given favorable "axioms of infinity". Parallelwise, for our system of extension, which builds upon a given initial system or ground level, the extension process naturally ends as the range of definable ordinals becomes exhausted.

So an extension system of our type is incomplete, which is fitting in relation to general undecidability results, but the extension process can be easily revived and renewed provided that new "axiomatic ordinals" are introduced. That is, it is always possible to add an axiom or axioms specifically directed to the theme of the existence of ordinal numbers so that it becomes possible to define a "new" ordinal level, with the validity of

the "orddef" associated with this level dependent on the new axiom or axioms, and such that the new definable ordinal is larger than all of those that were previously usable. And then the extension process is renewed and in the re-extended system the proof of formal consistency of the system as it was before becomes possible because the former range of the system is all subject to overview from the perspective of levels depending on the new axiom or axioms.

The interesting possibility is that it now becomes plausible or conceivable that the incompleteness phenomenon will not cause us to need to add anything like the Riemann Hypothesis or the Goldbach Conjecture as an axiom (because of the impossibility of otherwise establishing their truth) but rather that appropriate set theoretic axioms relevant to the existence of ordinal numbers may be added as needed.

Cultural Evolution

If it had been possible to achieve logical completeness with a system like Principia Mathematica then all of mathematical research to the extent of its not depending on tasteful definitions and inspired inventions of topics, and thus perhaps all "number theory", in the sense of the study of questions of classical elementary form, could be worked out straightforwardly by a robot operating in an isolation chamber simply on the basis of the rules.

But the history of human progress in science and mathematics reveals that observation of the phenomena of Nature has always played a large ro^le. Mathematics itself can be viewed as having evolved from the "language of precise quantitative communication" and many precise logical concepts have become included with the quantitative

concepts. The Sumerian scribes wrote down records for the quantities of grain received or sent from their central granaries.

So mathematical logic itself looks like a language that is naturally capable of evolution like also mathematics as a language and as an encyclopedia. But we feel that a "translatability" property should hold true here. Thus, for example, a relatively modern proof in geometry by Pascal should be translatable into a form, written in Greek, that Euclid would have found acceptable. Or the proof of the FLT by Wiles should be translatable into a version in French or Latin such that, with its introductory material developing the theory of elliptic curves and modular functions, it could have been studied and accepted by Fermat, if he could devote enough time to the effort.

In this context of the theme of cultural evolution, as applied specifically to mathematical logic, it seems possible that a system of logic of the future could be translated into a form corresponding to a system of the present time with the addition of a few axioms that give what is needed to give the potentialities of the future system. Our concept of an "hierarchical introspective logic" suggests that for "number theory" at least that there might be no need to adopt any new axioms other than essentially axioms of infinity that make it possible to assert the existence of ordinals that otherwise could not be thought of as definitely existing.

Already we have in set theory a variety of popular potential axioms. It is not yet generally assumed, but the "Axiom of Choice" is an extremely popular option in set theory. And if the adoption of conceivable or popular axioms went further then the GCH or "Generalized Continuum Hypothesis" could be adopted.

It seems plausible or at least conceivable

that knowledge actually gained from the study of Nature, plus cultural evolution, would in time lead to decisions, positive or negative, about the adoption of axioms relating to set theory that seem to us now as quite optional in merits.