CHAPTER 2

String Theory, 500 BCE

IT IS A STRANGE TRUISM: the earliest experimental science to establish quantitative relations between observable entities was acoustics. Pythagoras of Samos (ca. 585–500 BCE), the legendary philosopher who will forever be associated with the right-triangle theorem named after him, began his scientific career by investigating the vibrations of sound-emitting objects. According to legend, while walking down a street one day he heard sonorous sounds coming from a blacksmith's shop. Stopping by to investigate, he noticed that the sound had originated from the craftsman's hammer hitting a metal sheet; the heavier the sheet, the lower the pitch of the sound it emitted.

Not being satisfied with just a qualitative observation, Pythagoras went on to experiment with all kinds of vibrating bodies—taut strings, water-filled glasses, bells, and pipes (figure 2.1). He is said to have built a primitive musical instrument, the monochord—a single string attached to a sound board with a numerical scale along it (figure 2.2). The string's effective length could be varied by inserting a small bridge between the string and the board. Pythagoras found that, when the string was allowed to vibrate first at its full length and then stopped at half its length, the two sounds bore a pleasant, harmonious affinity to one another: they were separated by an *octave*. A melody played in different octaves sounds



FIGURE 2.1. Pythagoras experimenting with sound-emitting objects. From Franchino Gaffurio, *Theorica Musicae* (Milan, 1492).

essentially identical, like walking down the hallway on different floors of a hotel. The octave, Pythagoras had found, corresponds to the ratio 1:2.

Having established the octave as a fundamental musical interval, Pythagoras next attempted to subdivide this

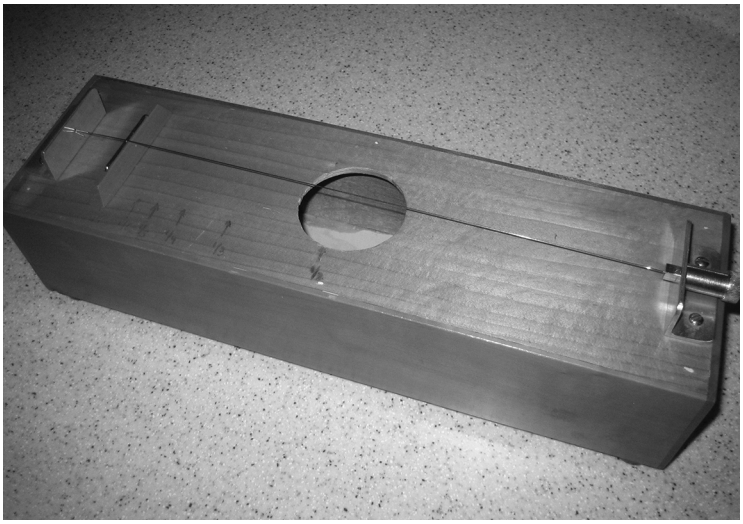


FIGURE 2.2. Monochord.

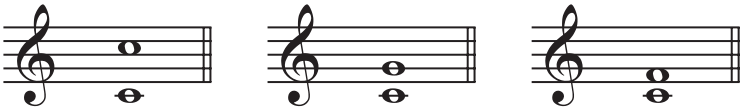


FIGURE 2.3. The octave, perfect fifth, and perfect fourth.

rather large interval into smaller parts. He experimented with other ratios of string length, leading him to a discovery that left a deep impression on him: ratios of small numbers produced harmonious, pleasant combinations of sounds—*consonances*—whereas ratios of larger numbers produced *dissonances*. Foremost among the former were the *octave* (1:2), the *fifth* (2:3), and the *fourth* (3:4) (the names derive from the position of these intervals in the musical scale; see figure 2.3). Pythagoras saw in this a sign that nature itself—indeed, the entire universe—is governed by simple numerical ratios. *Number rules the universe* became the Pythagorean motto, and it would dominate scientific thought for the next two thousand years.

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We must digress here for a moment and mention that beginning around 1600, it became the practice to describe musical intervals in terms of their *frequency ratios*, rather than ratios of string length. For any given string, the frequency is inversely proportional to the string's length, so the octave, the fifth, and the fourth correspond to the ratios 2:1, 3:2, and 4:3, respectively. We will adhere to this practice in what follows.

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The three intervals just mentioned were to play a fundamental role in music. Pythagoras called them *perfect consonances* and used them to construct a musical scale—the first known attempt to organize musical sounds into an orderly numerical system. He found that, starting with any note, going up a fifth and then another fourth brings us to a note exactly one octave above the starting note. Translated into ratios, the relation can be expressed as $\frac{3}{2} \times \frac{4}{3} = \frac{2}{1}$. This is true in general: *to add two intervals, multiply their frequency ratios*. Unbeknownst to him, Pythagoras had discovered the first logarithmic relation in history.

Next, he took each perfect consonance and raised its ratio to successive powers. Powers of 2:1 merely carry us to higher octaves, while powers of 4:3 result in inversions of 3:2 (an interval is said to be *inverted* if its lower note is moved up by one octave or its higher note down by one octave). This left him with powers of 3:2, starting with $(\frac{3}{2})^{-1} = \frac{2}{3}$ and leading to the following sequence:

$$\left(\frac{3}{2}\right)^{-1} \left(\frac{3}{2}\right)^{0} \left(\frac{3}{2}\right)^{1} \left(\frac{3}{2}\right)^{2} \left(\frac{3}{2}\right)^{3} \left(\frac{3}{2}\right)^{4} \left(\frac{3}{2}\right)^{5}$$
$$= \frac{2}{3} \quad 1 \quad \frac{3}{2} \quad \frac{9}{4} \quad \frac{27}{8} \quad \frac{81}{16} \quad \frac{243}{32}.$$

Of the seven ratios in this sequence, only the second and third lie within one octave. To bring the remaining ratios into the range of an octave, we multiply or divide them by powers of 2:

$$\frac{4}{3} \quad 1 \quad \frac{3}{2} \quad \frac{9}{8} \quad \frac{27}{16} \quad \frac{81}{64} \quad \frac{243}{128}.$$

When this new sequence is arranged in ascending order and augmented by the ratio 2:1 to complete it to a full octave, we get the following array:

$$1 \quad \frac{9}{8} \quad \frac{81}{64} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{27}{16} \quad \frac{243}{128} \quad \frac{2}{1}.$$

This sequence is known as a *diatonic scale*. It gives the ratio of each note to the fundamental (lowest) note. But in music, what matters most is the ratio *between* two notes, that is, the interval separating them. By taking the ratio of each note to the one preceding it, we get the sequence

$$\frac{9}{8} \quad \frac{9}{8} \quad \frac{256}{243} \quad \frac{9}{8} \quad \frac{9}{8} \quad \frac{9}{8} \quad \frac{256}{243},$$

which represents the intervals of the Pythagorean diatonic scale. It consists of just two distinct intervals, a large one of 9:8 (= 1.125), called a *whole tone*, and a small one of 256:243 (~ 1.053), called a *semitone* or *half tone*.

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At first thought, the Pythagorean scale seems like a great invention; it stands out for its simplicity, employing powers of just one ratio, 3:2. But this simplicity is deceiving, and for a number of reasons. First, as every music student learns early on, there is a scheme called a *circle of fifths*: start with any note and go up in a succession of fifths. After doing this twelve times (and in the process going through a series of sharps and flats, notes that are

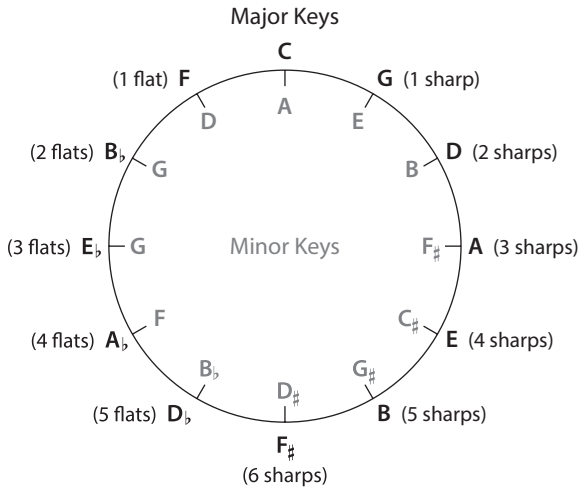


FIGURE 2.4. The circle of fifths.

a half tone above or below those of the diatonic scale), you should arrive back at the base note, albeit seven octaves higher (see figure 2.4). Alas, this is impossible to do with the Pythagorean scale: no positive integer values of m and n can ever satisfy the equation $(\frac{3}{2})^m = 2^n$.[1]

But even more troubling is the fact that the Pythagorean scale was out of tune with the natural sequence of *harmonics*, or *overtones*, generated by practically all musical instruments. When a string is vibrating, it emits a note with a definite pitch that can be placed on the musical staff, but there are also other, higher notes that come along with it. This mix of overtones gives the sound its characteristic color, or *timbre*—the quality that distinguishes the sound of a violin from that of a clarinet, even when they play the same note.

As we will see in the next chapter, the frequencies of these overtones are always whole multiples of the string's lowest, fundamental frequency, so they follow the sequence 1, 2, 3, . . . (relative to the fundamental). In theory

this series can go on forever, producing an infinite blend of ever higher notes. Usually, however, the amplitudes of these overtones, and therefore their intensities, quickly diminish as we go up the sequence, making them increasingly feeble and difficult to hear. Indeed, for nearly two thousand years they remained hidden behind the fundamental tone, barely noticed until the eighteenth century, when a little-known French scientist by the name Joseph Sauveur confirmed their existence (see chapter 3). Nevertheless, these harmonics play a crucial role in music, for they are the raw material from which the natural musical intervals are derived. The Pythagorean scale, being based solely on the ratio 3:2 while leaving out the remaining harmonics—including such important ratios as 5:4 and 6:5 (a major third and a minor third, respectively)—was therefore out of sync with the laws of acoustics; it was a purely mathematical creation, divorced from physical reality. This was the first known attempt to impose mathematical rules on music, but it would not be the last.

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The Pythagorean scale was typical of the Pythagorean philosophy in general. Obsession with musical numerology led Pythagoras's followers to believe that everything in the universe, from the laws of musical harmony to the motion of the celestial bodies, was governed by simple ratios of whole numbers. To understand this giant leap of faith, we must remember that in Greek tradition music ranked equal in status to arithmetic, geometry, and spherics (astronomy)—the *quadrivium* comprising the four disciplines every learned person was expected to master, the equivalent of the core curriculum of today's university.[2]

Significantly, to the Pythagoreans the word "arithmetic" had a different meaning than it has today; it meant *number theory*, the study of the properties of integers,

rather than the practical skills needed to compute with them. Likewise, they regarded the music component of the quadrivium as referring to *music theory*, the study of scales and harmony, not the actual art of playing music. This was typical of the aloof attitude of the Pythagoreans to all things practical. Theirs was a perfect universe, governed by notions of beauty, symmetry, and harmony but removed from daily, mundane considerations. It may have been one reason why they kept all their discussions secret, fearing they would be ridiculed by their fellow citizens, the vast majority of whom had to toil daily to eke out a living. None of the Pythagorean writings—if they left any writings at all—survived. All that we know about them came from later writers, who lived hundreds of years after Pythagoras and often outdid each other in extolling the virtues of their revered master.

But if their writings did not survive, the Pythagorean *legacy* lasted well over two thousand years. *Number rules the universe* became a rallying motto to generations of scientists and philosophers, who sought to explain the mysteries of the cosmos on the basis of musical ratios or in terms of simple, elegant geometric shapes. The planets, for example, had to move around the Earth in perfect circular orbits; it was inconceivable that any shape other than the perfectly symmetric circle could rule the universe. Thus, by subjugating the laws of nature to their ideals of beauty, harmony, and symmetry, the Pythagoreans may have actually impeded the progress of science for the next two millennia.

One of the last Pythagoreans was the eminent German astronomer Johannes Kepler (1571–1630), considered the father of modern astronomy. Kepler, at once a devout mystic and an ardent believer in the Copernican heliocentric system, spent more than half his life trying to derive the laws of planetary orbits from those of musical harmony.

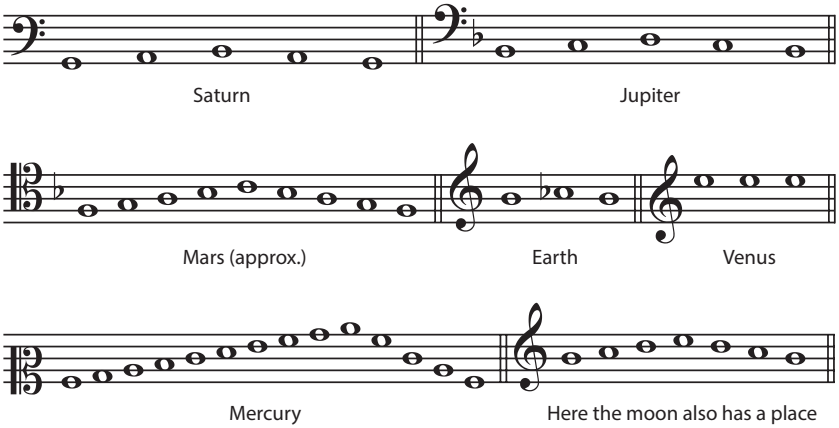


FIGURE 2.5. Kepler's planetary music. From *Harmonices Mundi*, Libri V (Linz: Jo Planck, 1619).

He believed that each planet, in its orbit around the Sun, plays a tune that our ears are unable to hear, being below the range of audible frequencies (not to mention that it was produced in the vacuum of outer space, where sound cannot propagate). He actually assigned a celestial melody, written down in musical notation, for each of the five then-known planets (figure 2.5)—the celebrated *music of the spheres*. It was only after decades, during which he followed this blind path, that Kepler finally abandoned the Greek circular orbits in favor of ellipses, to which Newton, a generation later, would add the parabola and hyperbola.

NOTES

1. This can be seen by rewriting the equation as $3^m = 2^k$, where $k = m + n$. Now the left side of this equation is a power of 3 and is therefore an odd integer, while the right side is a power of 2 and thus even.
2. The term *quadrivium* is attributed to Boethius (sixth century CE), but the curriculum it embodied was already outlined in Plato's *The Republic*. Together with the *trivium* (grammar, logic, and rhetoric), it comprised the seven liberal arts of medieval universities.