H. B. CURRY. *A note on the associative law in logical algebras.* ***Bulletin of the American Mathematical Society,*** vol. 42 (1936), pp. 523–524.

A generalization of Bernays' proof of the redundancy of the associative law in Part I Section A of ***Principia mathematica,*** showing that any system, containing a binary operation (denoted by juxtaposition) and a relation, $<$, permitting inference, such that (1) $p<pp$, (2) $pq<p$, (3) $pq<qp$, (4) *If* $p<q$ *and* $q<r$, *then* $p<r$, and either (5) *If* $p<q$, *then* $rp<rq$, or (6) *If* $p<q$ *and* $p<r$, *then* $p<qr$, must also contain all forms of the associative law.

PAUL HENLE

V. C. ALDRICH. *Renegade instances.* ***Philosophy of science,*** vol. 3 (1936), pp. 506–514.

If the sentence, "Swans are birds," stand for an empirical generalization, and is therefore synthetic, it may be invalidated by negative instances. If, however, it stand for an analytic proposition—the intension of swan including the property of being a bird—then, since negative instances are impossible, the proposition cannot be invalidated by one. Nonetheless we may find instances having the defining properties of swan except part of the intension of bird (e.g., biped). A sufficient number of such, or a single impressive one, may render our definition of swan inconvenient, thereby resulting in our redefining the word. Such instances Mr. Aldrich calls "renegade." Whether such an instance does "renegade" the *a priori* or definitional generalization depends wholly upon pragmatic considerations. The author presents no discussion of these considerations, of their bearing on scientific classification, or of the origin of such *a priori* generalizations in the empirical sciences.

EVERETT J. NELSON

RUDOLF CARNAP and FRIEDRICH BACHMANN. *Über Extremalaxiome.* ***Erkenntnis,*** vol. 6 (1936), pp. 166–188.

The authors discuss axiom systems of the following sort. Superposed upon a finite sequence of axioms (the "Rumpfsystem") each of which makes a certain assertion with regard to the fundamental concepts employed, appears a final axiom seemingly concerning the preceding axioms and not related to the fundamental concepts of the system. The best known such axiom-system is that of Hilbert for Euclidean geometry, with its famous "Axiom of Completeness." Whether the final axiom states that no more inclusive system of things exists which satisfies the preceding—and is therefore a "maximal" axiom—or is analogously a "minimal" axiom, such a final axiom will be called an "extremal" axiom. The authors defend the use of such axioms under suitable restrictions and when properly stated and interpreted. A fundamental concept in the study of axiomatics is the notion of isomorphism which the authors extend, by the concept of *correlators* which are binary relations between given n-ary relations. Complete isomorphism is discussed with respect to types of like specified order. If any two structures satisfying the "Rumpfsystem" are completely isomorphic as to elements of specified order, one may then inquire as to whether such a structure does or does not have a proper substructure isomorphic with it. Distinction is made between extensions of model and extension of structure. The legitimate introduction of the extremal axiom corresponds to the selection of extremal structures. The question of independence of the axioms in the "Rumpfsystem" as affected by the introduction of an extremal postulate is discussed and various cases are found to occur. A final serious question arises with regard to extension to a system of different order-type, as occurs from the system of rational numbers to that of real numbers regarded as sequences of rationals. Tarski's restriction to an increase of one unit in order type has many attractive features, and avoids certain serious difficulties, but is found to be somewhat too restrictive.

ALBERT A. BENNETT

A. M. TURING. *On computable numbers, with an application to the Entscheidungsproblem.* ***Proceedings of the London Mathematical Society,*** 2 s. vol. 42 (1936–7), pp. 230–265.

The author proposes as a criterion that an infinite sequence of digits 0 and 1 be "computable" that it shall be possible to devise a computing machine, occupying a finite space and with working parts of finite size, which will write down the sequence to any desired number of terms if allowed to run for a sufficiently long time. As a matter of convenience, certain further restrictions are imposed on the character of the machine, but these are of such a nature as obviously to cause no loss of generality—in particular, a human calculator, provided with pencil and paper and explicit in-

structions, can be regarded as a kind of Turing machine. It is thus immediately clear that computability, so defined, can be identified with (especially, is no less general than) the notion of effectiveness as it appears in certain mathematical problems (various forms of the Entscheidungsproblem, various problems to find complete sets of invariants in topology, group theory, etc., and in general any problem which concerns the discovery of an algorithm).

The principal result is that there exist sequences (well-defined on classical grounds) which are not computable. In particular the *deducibility problem* of the functional calculus of first order (Hilbert and Ackermann's engere Funktionenkalkül) is unsolvable in the sense that, if the formulas of this calculus are enumerated in a straightforward manner, the sequence whose nth term is 0 or 1, according as the nth formula in the enumeration is or is not deducible, is not computable. (The proof here requires some correction in matters of detail.)

In an appendix the author sketches a proof of equivalence of "computability" in his sense and "effective calculability" in the sense of the present reviewer (***American journal of mathematics,*** vol. 58 (1936), pp. 345–363, see review in this JOURNAL, vol. 1, pp. 73–74). The author's result concerning the existence of uncomputable sequences was also anticipated, in terms of effective calculability, in the cited paper. His work was, however, done independently, being nearly complete and known in substance to a number of persons at the time that the paper appeared.

As a matter of fact, there is involved here the equivalence of three different notions: computability by a Turing machine, general recursiveness in the sense of Herbrand-Gödel-Kleene, and λ-definability in the sense of Kleene and the present reviewer. Of these, the first has the advantage of making the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately—i.e. without the necessity of proving preliminary theorems. The second and third have the advantage of suitability for embodiment in a system of symbolic logic.

ALONZO CHURCH

EMIL L. POST. *Finite combinatory processes—formulation 1.* ***The journal of symbolic logic,*** vol. 1 (1936), pp. 103–105.

The author proposes a definition of "finite 1-process" which is similar in formulation, and in fact equivalent, to computation by a Turing machine (see the preceding review). He does not, however, regard his formulation as certainly to be identified with effectiveness in the ordinary sense, but takes this identification as a "working hypothesis" in need of continual verification. To this the reviewer would object that effectiveness in the ordinary sense has not been given an exact definition, and hence the working hypothesis in question has not an exact meaning. To define effectiveness as computability by an arbitrary machine, subject to restrictions of finiteness, would seem to be an adequate representation of the ordinary notion, and if this is done the need for a working hypothesis disappears.

The present paper was written independently of Turing's, which was at the time in press but had not yet appeared.

ALONZO CHURCH

H. B. SMITH. *The algebra of propositions.* ***Philosophy of science,*** vol. 3 (1936), pp. 551–578.

The author is proposing a calculus of propositions based on four primitive ideas: disjunction $p+q$, conjunction pq, negation p', and implication $p \angle q$. Although not explicitly stated, it is apparently intended that the first three operations shall obey all the usual laws. The implication $p \angle q$ is not, however, to be identified with $p'+q$, and is thus in some degree analogous to C. I. Lewis's $p \dashv q$. A modal operation $|p|$, analogous to Lewis's $\Diamond p$, is defined as $(p \angle 0)'$, where 0 is the null-proposition (a proposition q such that $q \angle q'$). Equivalence is expressed by $p=q$, apparently to be defined as $(p \angle q)(q \angle p)$.

On intuitive grounds not entirely clear, the author requires that "all modal distinctions" shall be recognized. That is, let two expressions be formed from the letter p, each by a finite number of applications of negation and the modal operation, negation being nowhere applied twice in succession (i.e. without one or more intervening applications of the modal operation); then these two expressions shall not be assumed equivalent unless they are identical *expressions.*

An immediate difficulty is that if we assume (1) $p \angle ||p|'|'$ and (2) $(p \angle q)(q \angle r) \angle (p \angle r)$ and the principle of inference (3), "If P and PQ∠R then Q∠R," then it is possible to infer $|p|' = |||p|'|'|'$. This the author meets by rejecting (2). (In connection with an earlier note on this same point, the