A New Approach to Proving the Pythagorean Theorem

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Abstract

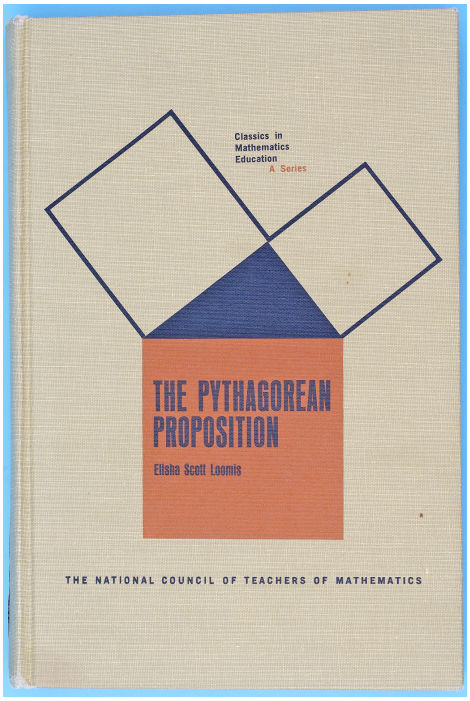
Ne'Kiya D. Jackson and Calcea Rujean Johnson presented a trigonometric proof of the Pythagorean Theorem at the 2023 AMS Spring Southeastern Sectional Meeting claiming that it is an *impossible* proof. They cited a false claim in Loomis' 1907 book "*There are no trigonometric proofs. ..... Trigonometry is because the Pythagorean Proposition is*." This note presents a new method based on similarity and geometric progression with which a pure geometrical proof is given. Additionally, this note also discusses some proofs in Loomis' book and provides more new proofs using the concept of the Lemoine Point. Finally, the Appendix has Zimba's original proof of the angle difference identities for sin() and cos() without using the Pythagorean Theorem. It also includes a proof of the angle sum identities for sin() and cos() without using the Pythagorean Theorem. Both results can be used to prove the Pythagorean Identity. As a result, Loomis' claim is false and the proof of Jackson-Johnson can be quickly replaced by a purely geometrical one.

A proof of the Pythagorean Theorem using trigonometry was presented at the AMS Spring Southeastern Sectional Meeting on March 18, 2023 by Ne'Kiya D. Jackson and Calcea Rujean Johnson [4]. This was reported widely by the media such as *The Guardian* [12], *Popular Mechanics* [8] and *Scientific American* [11]. Unfortunately, the authors and some reports kept suggesting that a trigonometric proof is "impossible." They all cited a 1907 book *The Pythagorean Proposition* by Elisha Scott Loomis [6, second edition, pp. 244-245] (Figure 1(a)) in which Loomis (Figure 1(b)) stated the following:

> *Facing forward the thoughtful reader may raise the question: Are there any proofs based upon the science of trigonometry or analytical geometry?*
>
> *There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem; because of this theorem we say* $\sin^2 A + \cos^2 A = 1$*, etc. Trigonometry is because the Pythagorean Theorem is* [6, p.244].

This is false, because Zimba [13] showed that the validity of $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ and $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$ is independent of the *Pythagorean Identity*[1] which can be proved using $\cos(\alpha-\beta)$ by setting $\alpha = \beta$. This author is unaware of other

[1]The *Pythagorean Identity* refers to the identity $\sin^2(x) + \cos^2(x) = 1$, which implies the Pythagorean Theorem immediately. Conversely, if the Pythagorean Identity holds, the Pythagorean Theorem can be obtained easily.



(a) The 1940 Second Edition Published by The National Council of Teachers of Mathematics



(b) Elisha Scott Loomis (Photograph Taken 1935)

Figure 1

publications between Loomis' book and Zimba's paper that demonstrated similar results; however, Zimba's work proved that Loomis' claim indeed is false.

We will develop a simple method based on similarity and geometric progression to prove the Pythagorean Theorem. While this method can be applied to more general geometric shapes, we only focus on right triangles. In what follows, Section 1 presents our method; Section 2 shows that some classical proofs in Loomis' book can easily be converted to use this technique, Section 3 presents Jackson and Johnson's proof without using trigonometry; Section 4 discusses the original trigonometrical version; Section 5 first discusses the concepts of symmedians and the Lemoine point of a triangle, and then proceeds to offer three more proofs based on the Lemoine point, and Section 6 has our conclusions. Section 2 is divided into three subsections: Section 2.1 discusses proofs in which square dissection is used, Section 2.2 has simple proofs that use the given right triangle directly, and Section 2.3 includes a proof which has a square on the hypothenuse. Finally, the Appendix includes Zimba's proof showing that the angle difference identities for $\sin()$ and $\cos()$ can be derived without using the Pythagorean Identity. Following Zimba's idea, we show that the angle sum identities for $\sin()$ and $\cos()$ are also independent of the Pythagorean Identity. Furthermore, from the angle sum identities the sum-to-product identities are derived from which

the derivatives of sin() and cos() are shown to be independent of the Pythagorean Identity. Then, we use the double angle identities and L'Hôpital's Rule to prove the Pythagorean Identity. Finally, because the derivative computation of sin() and cos() are independent of the Pythagorean Identity, a simple application of calculus yields the identity. Therefore, this chain of reasoning suggests that "*Trigonometry is because the Pythagorean Theorem is*" is false.

1 The Main Idea

Given a polygonal shape A and a polygonal shape $\mathsf{B} \subseteq \mathsf{A}$, if B is similar to A with a scaling factor ρ (*i.e.*, any edge q of B and its corresponding edge p of A satisfying $q = \rho \cdot p$, where $0 < \rho < 1$), then $\mathcal{A}(\mathsf{A}) = \mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B})$, where $\mathcal{A}(\mathsf{X})$ denotes the area of X (Figure 2).

Note that a scaling factor ρ for length induces a scaling factor ρ^2 for area. Because $\mathsf{B} \sim \mathsf{A}$, $\mathcal{A}(\mathsf{B}) = \rho^2\mathcal{A}(\mathsf{A})$, we have $\mathcal{A}(\mathsf{A}) = \mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B}) = \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A})$. Therefore, we have

$$
\begin{aligned}
\mathcal{A}(\mathsf{A}) &= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2(\mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{B})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4(\mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{B}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4(\rho^2\mathcal{A}(\mathsf{A})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^6\mathcal{A}(\mathsf{A}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^6(\mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^6\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^6\mathcal{A}(\mathsf{B}) \\
&\vdots \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B})\left(1 + \rho^2 + \rho^4 + \rho^6 + \cdots\right) \\
&= \frac{\mathcal{A}(\mathsf{A}-\mathsf{B})}{1-\rho^2}
\end{aligned}
\tag{1}
$$

Hence, if we are able to find B and ρ and compute $\mathcal{A}(\mathsf{A}-\mathsf{B})$, it is easy to find $\mathcal{A}(\mathsf{A})$.

As for line segment length, the scaling factor is only ρ. If a point Z is selected on a line segment XY, we have of $\overline{XY} = \overline{XZ} + \overline{ZY}$ (Figure 3). Let $\rho = \overline{ZY}/\overline{XY}$. Based on the idea above we have

$$
\overline{XY} = \overline{XZ} + \rho\overline{XZ} + \rho^2\overline{XZ} + \rho^3\overline{XZ} + \rho^4\overline{XZ} + \cdots = \frac{\overline{XZ}}{1-\rho} \tag{2}
$$

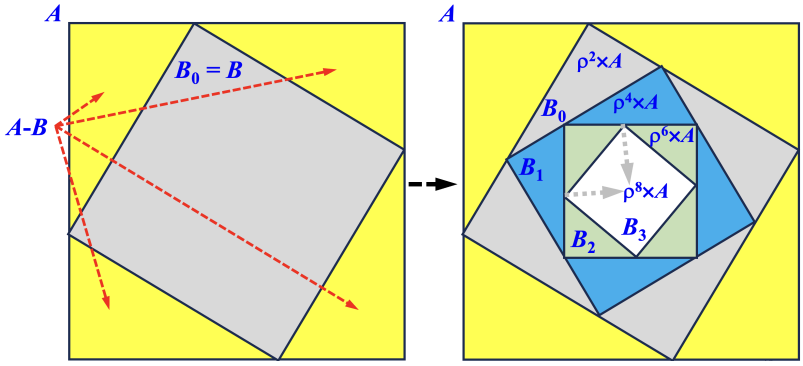


Figure 2: This Is How the Idea Goes for Area Computation

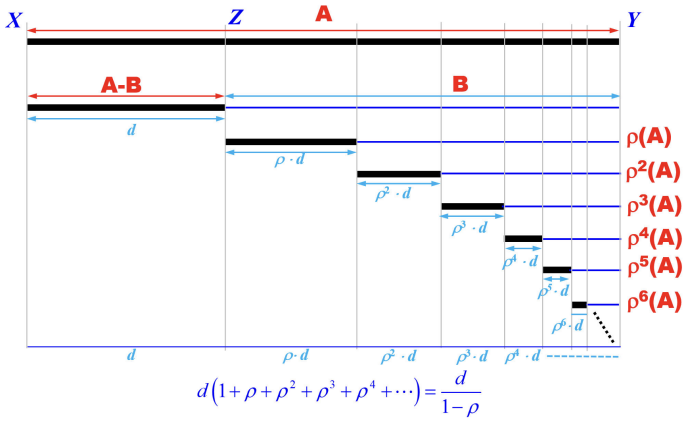


Figure 3: This Is How the Idea Goes for Line Segment Length Computation

2 Re-Do Some Classical Proofs

Many proofs in Loomis' book [6] can easily be redone with the new method. The next few sub-sections discuss how this conversion can be done easily. First, a shape A is constructed from the given right triangle of sides $a \leq b < c$ with c being the hypotenuse. Second, find a sub-shape B that is similar to A and the area of $\mathsf{A} - \mathsf{B}$ can be computed easily. Third, find the scaling factor ρ. Fourth, use our method to compute the area of figure A. Fifth, find another way to compute the area of A without using B. Finally, equating the two results followed by some simplification yields the desired result. However, we have to point out that for the Pythagorean Theorem, the length of the hypotenuse c should be used in the first stage and should not be cancelled out because c is typically not used in the second stage.

2.1 Proofs Involving the Use of a Square

Proof 1

In Figure 4(a) the square has side length $a+b$ and the right triangle is repeated four times inside the square. The scaling factor going from the outer square to the inner one is $\rho = c/(a+b)$ and hence we have

$$\frac{1}{1-\rho^2} = \frac{(a+b)^2}{(a+b)^2 - c^2}$$

Therefore, the area of the square is

$$(a+b)^2 = \frac{1}{1-\rho^2} \times 4\left(\frac{1}{2}a \cdot b\right) = \frac{(a+b)^2}{(a+b)^2 - c^2} \times 4\left(\frac{1}{2}a \cdot b\right)$$

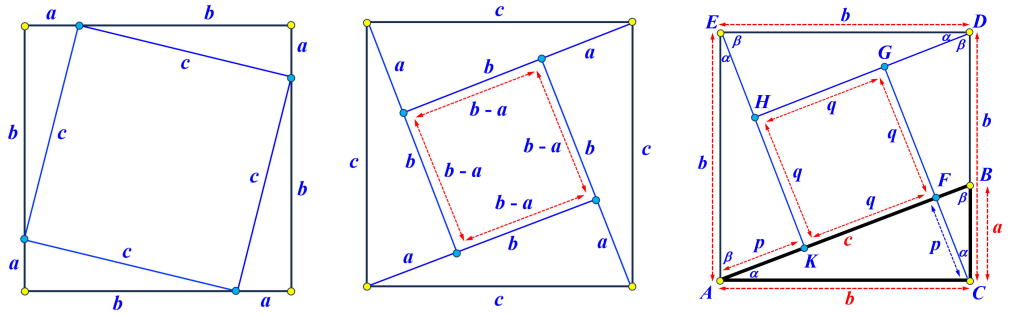
After simplifying the above, we get $a^2 + b^2 = c^2$ (Loomis [6, Proof Thirty-Three, p. 48]).

Proof 2

Figure 4(b) is another commonly seen proof in which the inner square has side length $b-a$. The scaling factor is $\rho = (b-a)/c$ and $1/(1-\rho^2) = c^2/(c^2-(b-a)^2)$. The area of the square is computed as follows:

$$c^2 = \frac{1}{1-\rho^2} \times 4\left(\frac{1}{2}a \cdot b\right) = \frac{c^2}{c^2 - (b-a)^2} \times 4\left(\frac{1}{2}a \cdot b\right)$$

Again, simplifying the above yields $a^2 + b^2 = c^2$.



(a) (b) (c)

Figure 4: Proofs Involving the Use of Squares

Proof 3

In Loomis [6, Proof Two Hundred Fifteen, p. 221] a proof similar to the above one is shown. Given a right triangle $\triangle ABC$ (Figure 4(c)), construct a square of side length b on side AC and drop

a penpendicular from C to $\overleftrightarrow{AB}$ meeting it at F. Then, drop a perpendicular from D to $\overleftrightarrow{CF}$ meeting it at G and perpendiculars from E to $\overleftrightarrow{DG}$ and $\overleftrightarrow{AB}$ meeting them at H and K, respectively. It is obvious that $\triangle AFC \cong \triangle CGD \cong \triangle DHE \cong \triangle EKA$ and the lengths of AK and CF are equal. Furthermore, because $\triangle ABC \sim \triangle ACF$, we have

$$p = \overline{AK} = \frac{a \cdot b}{c}, \qquad p+q = \overline{AF} = \frac{b^2}{c} \quad \text{and} \quad q = \overline{KF} = \overline{AF} - \overline{AK} = \frac{b(b-a)}{c}$$

Hence, the sum of the areas of the four right triangles is

$$4\left(\frac{1}{2}p(p+q)\right) = \frac{1}{2}\frac{ab^3}{c^2} \tag{3}$$

The scaling factor going from square $ACDE$ to square $FGHK$ is

$$\rho = \frac{q}{b} = \frac{(b/c)(b-a)}{c} = \frac{b-a}{c} \quad \text{and} \quad \frac{1}{1-\rho^2} = \frac{c^2}{c^2-(b-a)^2}$$

The area of the outer square is

$$\mathcal{A}(ACDE) = \frac{1}{1-\rho^2} \times \left(\frac{1}{2}\frac{ab^3}{c^2}\right) = \frac{c^2}{c^2-(b-a)^2} \times \left(\frac{1}{2}\frac{ab^3}{c^2}\right) = \frac{2ab^3}{c^2-(b-a)^2}$$

Because $\mathcal{A}(ACDE) = b^2$ which is equal to the above result, we have

$$b^2 = \frac{2ab^3}{c^2-(b-a)^2}$$

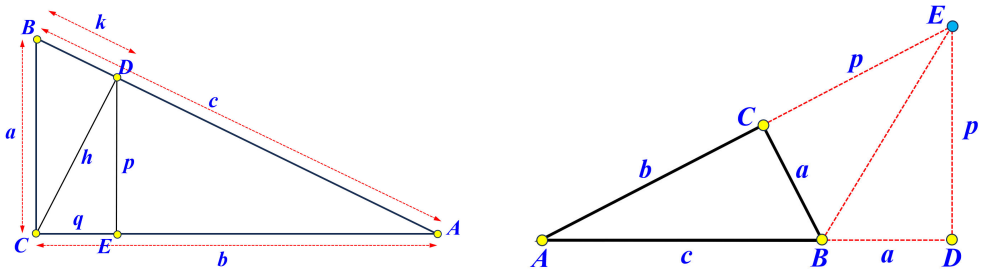
Simplifying yields $c^2 = a^2 + b^2$.

Notes

Some proofs in Loomis [6] share the same technique, although the division of the sides of the square may not be $a : b$. For example, in Loomis [6, Proof Sixty-Three, p. 137], the division of the side c square is exactly $a : b$; but other rectangles and squares are needed to complete the proof. Loomis [6, Proof Thirty-Three, p.48] is exactly the same as shown in Figure 4(b). Loomis [6, Proof One Hundred Thirty-Three, p. 177] is similar to Figure 4(c), but the division of side c is $ac/b : c(b-a)/b$. Other proofs in Loomis [6] are similar (*e.g.*, Proofs 131–132, Proofs 134–137, etc.) and use different ways of cutting the square of side c. These proofs can also be transformed to use the technique presented here.

2.2 Proofs Based on Similar Right Triangles Inside or Outside the Given One

Proof 4



(a) (b)

Figure 5: Very Simple Proofs Using the Given Triangle Directly

Consider $\triangle ABC$ in Figure 5(a), where D is the perpendicular foot from C to side $\overleftrightarrow{AB}$. Line $\overleftrightarrow{CD}$ divides $\triangle ABC$ into two smaller triangles both similar to $\triangle ABC$ (Loomis [6, Proof One, p. 23]).

From $\triangle CDB \sim \triangle ACB$, we have $h = (a \cdot b)/c$ and $k = a^2/c$. Therefore, we have

$$\mathcal{A}(\triangle CBD) = \frac{1}{2} \cdot h \cdot k = \frac{1}{2}(a \cdot b)\left(\frac{a}{c}\right)^2 \tag{4}$$

Because $\triangle ACD \sim \triangle ABC$, the scaling factor ρ from $\triangle ABC$ to $\triangle ACD$ is $\rho = h/c = b/c$. Hence, we have

$$\mathcal{A}(\triangle ABC) = \frac{\mathcal{A}(\triangle CBD)}{1-\rho^2} = \frac{1}{2}\frac{a^3 b}{c^2 - b^2} \tag{5}$$

Because we also have $\mathcal{A}(\triangle ABC) = (a \cdot b)/2$, the following holds:

$$\frac{1}{2} a \cdot b = \frac{1}{2}\frac{a^3 b}{c^2 - b^2}$$

Simplifying the above yields

$$1 = \frac{a^2}{c^2 - b^2}$$

This leads to $c^2 = a^2 + b^2$, the desired result.

Proof 5

As a direct consequence of Proof 5, a very similar one was discussed in the *Cut the Knot* site [1], credited to John Arioni. From D drop a perpendicular to $\overleftrightarrow{AC}$ meeting it at E (Figure 5(a)). Let $p = \overline{DE}$ and $q = \overline{CE}$. Because $\triangle DCE \sim \triangle ABC$, we have

$$\frac{p}{h} = \frac{b}{c} \quad \text{and} \quad \frac{q}{h} = \frac{a}{c}$$

Because we know $h = (a \cdot b)/c$, we have

$$p = \frac{a \cdot b^2}{c^2} \quad \text{and} \quad q = \frac{a^2 \cdot b}{c^2}$$

The area of trapezoid $BCED$ is:

$$\mathcal{A}(BCED) = \frac{1}{2}(p+a)\cdot q = \frac{a^3 b}{2c^4}(b^2+c^2) \tag{6}$$

The scaling factor going from $\triangle ABC$ to $\triangle ADE$ is $\rho = p/a = (b/c)^2$ and

$$\frac{1}{1-\rho^2} = \frac{c^4}{(c^2-b^2)(c^2+b^2)}$$

Then, the area of $\triangle ABC$ is

$$\mathcal{A}(ABC) = \frac{1}{1-\rho^2} \times \left(\frac{a^3 b}{2c^4}(b^2+c^2)\right) = \frac{c^4}{(c^2-b^2)(c^2+b^2)} \times \left(\frac{a^3 b}{2c^4}(b^2+c^2)\right) = \frac{a^3 b}{2(c^2-b^2)}$$

However, because $\mathcal{A}(ABC) = (a\cdot b)/2$, we have

$$\frac{1}{2}(a\cdot b) = \frac{a^3 b}{2(c^2-b^2)}$$

Again, we have $c^2 = a^2 + b^2$. Note that this proof is essentially applying the previous proof twice, once reducing $\triangle ABC$ to $\triangle ACD$ and the other reducing $\triangle ACD$ to $\triangle ADE$.

Proof 6

Proofs Three and Four in Loomis [6, p. 26] share the same idea as discussed in the first proof in this section. We only discuss Proof Four here and Proof Three can be obtained exactly the same way. In Figure 5(b), $\triangle ABC$ is the given right triangle. Extend the hypotenuse $\overleftrightarrow{AB}$ to D so that $\overline{BD} = \overline{BC} = a$, and construct a line $\overleftrightarrow{DE}$ perpendicular to $\overleftrightarrow{AB}$ meeting $\overleftrightarrow{AC}$ at E. It is obvious that $\triangle BDE \cong \triangle BCE$ and $\triangle AED \sim \triangle ABC$. As a result, we have $p = (a/b)(a+c)$. The area of quadrilateral $BCED$ is

$$\mathcal{A}(BCED) = 2\left(\frac{1}{2}a\cdot p\right) = \frac{a^2(a+c)}{b}$$

The scaling factor ρ bringing $\triangle AED$ to $\triangle ABC$ is

$$\rho = \frac{a}{p} = \frac{b}{a+c} \qquad \text{and} \qquad \frac{1}{1-\rho^2} = \frac{(a+c)^2}{(a+c)^2 - b^2}$$

Consequently, we have

$$\mathcal{A}(AED) = \frac{1}{1-\rho^2}\mathcal{A}(BCED) = \left(\frac{(a+c)^2}{(a+c)^2-b^2}\right)\left(\frac{a^2(a+c)}{b}\right) = \frac{a^2(a+c)^3}{b\left((a+c)^2-b^2\right)}$$

However. $\mathcal{A}(AED)$ may also be calculated as

$$\mathcal{A}(AED) = \frac{1}{2}p(a+c) = \frac{1}{2}\frac{a(a+c)^2}{b}$$

Both results must agree:

$$\frac{a^2(a+c)^3}{b\,((a+c)^2-b^2)} = \frac{1}{2}\frac{a(a+c)^2}{b}$$

A simple simplification yields $c^2 = a^2 + b^2$.

2.3 Proofs with Squares Standing on the Sides of a Right Triangle

Proof 7

There are many proofs in which a square is constructed on a side of a right triangle, and some of these proofs can easily be adapted for our method. The following is taken from Loomis [6, Proof Nineteen, p. 43] (Figure 6). Because of $\triangle A_1AA_2 \sim \triangle ABC$ and $\triangle BB_1B_2 \sim \triangle ABC$, we have $p = (bc)/a$ and $q = (ca)/b$. As a result, the length of side $\overline{A_1B_1}$ is:

$$\overline{A_1B_1} = p + c + q = \frac{c}{ab}(ab + a^2 + b^2)$$

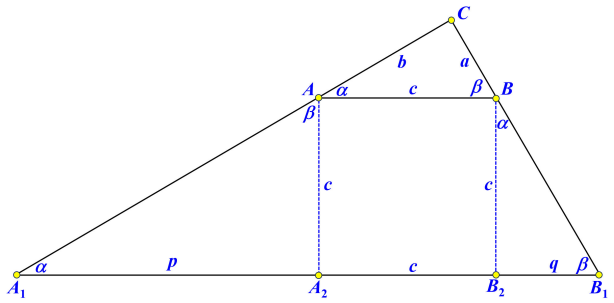


Figure 6: A Square on Side c (*i.e.*, $\overline{AB}$)

The area of the trapezoid ABB_1A_1 is

$$\mathcal{A}(AA_1B_1B) = \frac{1}{2}(c + \overline{A_1B_1}) \cdot c = \frac{c^2}{2} \cdot \frac{(a+b)^2}{ab}$$

The scaling factor ρ is the ratio of c and $\overline{A_1B_1}$

$$\rho = \frac{c}{\overline{A_1B_1}} = \frac{ab}{ab + a^2 + b^2}$$

Therefore, we have

$$\rho^2 = \frac{(ab)^2}{(ab + a^2 + b^2)^2} \qquad \text{and} \qquad \frac{1}{1-\rho^2} = \frac{(ab + a^2 + b^2)^2}{(a^2 + b^2)(a+b)^2}$$

Hence, the area of $\triangle CA_1B_1$ is calculated from the area of the trapezoid ABB_1A_1 as follows:

$$\mathcal{A}(CA_1B_1) = \frac{1}{1-\rho^2}\mathcal{A}(ABB_1A_1) = \frac{(ab+a^2+b^2)^2}{(a^2+b^2)(a+b)^2}\cdot\frac{c^2}{2}\frac{(a+b)^2}{ab} = \frac{1}{2}\cdot\frac{c^2(ab+a^2+b^2)^2}{ab(a^2+b^2)} \quad (7)$$

Because of similarity, the lengths of side CA_1 and CB_1 are simply $\overline{CA_1} = b/\rho$ and $\overline{CB_1} = a/\rho$. Consequently, the area of $\triangle CA_1B_1$ is also calculated as follows:

$$\mathcal{A}(CA_1B_1) = \frac{1}{2}\overline{CA_1}\cdot\overline{CB_1} = \frac{1}{2}\cdot\frac{a}{\rho}\cdot\frac{b}{\rho} = \frac{1}{2}\cdot\frac{(ab+a^2+b^2)^2}{ab}$$

This result must agree with the one in Eqn. (7):

$$\frac{1}{2}\cdot\frac{c^2(ab+a^2+b^2)^2}{ab(a^2+b^2)} = \frac{1}{2}\cdot\frac{(ab+a^2+b^2)^2}{ab}$$

Simplifying the above yields $c^2 = a^2+b^2$.

It does not have to use area in this particular case. Because $\triangle ABC \sim \triangle A_1B_1C$, the altitude from from C to $\overleftrightarrow{AB}$ and the altitude from C to $\overleftrightarrow{A_1B_1}$ are $h = (a\cdot b)/c$ and $h+c = (a\cdot b+c^2)/c$, respectively, and hence the scaling factor $\overline{\rho}$ can also be computed as follows:

$$\overline{\rho} = \frac{h}{h+c} = \frac{ab}{ab+c^2}$$

Because $\rho = \overline{\rho}$, $c^2 = a^2+b^2$ follows immediately.

3 Jackson and Johnson's Proof without Trigonometry

Proof 8

This section will re-do the proof of Ne'Kiya D. Jackson and Calcea Rujean Johnson. The construction is similar, but the proof is completely geometrical without the use of trigonometry. Given a right triangle $\triangle ABC$ with $\angle A = \alpha$, $\angle B = \beta > \alpha$, $\angle C = 90^\circ$, $a = \overline{BC}$, $b = \overline{CA}$ and $c = \overline{AB}$ (Figure 8). The case of $\alpha = \beta = 45^\circ$ will be addressed separately. Given a line segment $\overline{Y_0Z_0}$ of length x, construct a triangle $\triangle XY_0Z_0$ so that $\angle Y_0 = \alpha$ and $\angle Z_0 = \alpha+90^\circ$. Note that this construction does not work if $\alpha = \beta = 45^\circ$ because X is at infinity and the area of $\triangle XY_0Z_0$ is not finite. From Z_0 construct a line perpendicular to $\overleftrightarrow{Y_0Z_0}$ meeting $\overleftrightarrow{XY_0}$ at Y_1 and then construct a line perpendicular to $\overleftrightarrow{Y_1Z_0}$ at Y_1 meeting $\overleftrightarrow{XZ_0}$ at Z_1. Let $p = \overline{Y_0Y_1}$, $q = \overline{Z_0Z_1}$, $r = \overline{Y_1Z_1}$ and $h = \overline{Z_0Y_1}$.

Because $\triangle ABC \sim \triangle Y_0Y_1Z_0$, we have $h = x(a/b)$ and $p = x(c/b)$. Because $\triangle ABC \sim \triangle Z_0Z_1Y_1$, we have $q = h(c/b) = x(ac/b^2)$ and $r = x(a/b)^2$. As a result, the area of trapezoid $Y_0Z_0Z_1Y_1$ is

$$\mathcal{A}(Y_0Z_0Z_1Y_1) = \frac{1}{2}(x+r)\cdot h = \frac{1}{2}\left[x+x\left(\frac{a}{b}\right)^2\right]\cdot\left(\frac{xa}{b}\right) = \frac{x^2}{2}\frac{a(a^2+b^2)}{b^3} \quad (8)$$

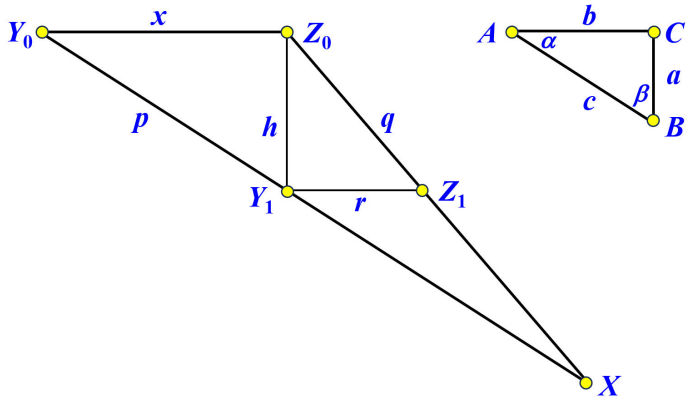


Figure 7

Figure 8: Jackson and Johnson's Proof: Part 1

Because $\triangle XY_0Z_0 \sim \triangle XY_1Z_1$ and $r/x = (a/b)^2$, the scaling factor from $\triangle XY_0Z_0$ to $\triangle XY_1Z_1$ is $\rho = (a/b)^2$. Hence, we have

$$\rho^2 = \left(\frac{a}{b}\right)^4 \quad \text{and} \quad \frac{1}{1-\rho^2} = \frac{b^4}{b^4-a^4} = \frac{b^4}{(b^2-a^2)(b^2+a^2)}$$

The area of $\triangle XY_0Z_0$ is:

$$\mathcal{A}(\triangle XY_0Z_0) = \frac{1}{1-\rho^2}\cdot\mathcal{A}(Y_0Z_0Z_1Y_1) = \left(\frac{b^4}{(b^2-a^2)(b^2+a^2)}\right)\left(\frac{x^2}{2}\frac{a(a^2+b^2)}{b^3}\right) = \frac{x^2}{2}\frac{ab}{b^2-a^2} \quad (9)$$

Then, we determine the lengths of $\overline{XY_0}$ and $\overline{XZ_0}$. Because we know $\overline{Y_0Y_1} = p = x(c/b)$ and $\rho = (a/b)^2$, our method (Eqn. (2)) yields

$$\overline{XY_0} = \frac{p}{1-\rho} = x\frac{bc}{b^2-a^2} \quad \text{and} \quad \overline{XZ_0} = \frac{q}{1-\rho} = x\frac{ac}{b^2-a^2} \quad (10)$$

Construct a line perpendicular to $\overleftrightarrow{XY_0}$ at Y_0 meeting $\overleftrightarrow{XZ_0}$ at X'. It is not difficult to see that $\triangle X'Y_0Z_0$ is an isoceles with $\angle Y_0 = \angle Z_0 = \beta$ and $\angle X' = 2\alpha$ (Figure 9).

The length k of the altitude on side $\overline{Y_0Z_0}$ is $k = (x/2)(b/a)$, and we have

$$\mathcal{A}(\triangle X'Y_0Z_0) = \frac{1}{2}(x\cdot k) = \frac{x^2}{2^2}\left(\frac{b}{a}\right) \quad (11)$$

The length t of side $\overline{X'Y_0}$ is $t = (x/2)(c/a)$. Therefore, the area of triangle $\triangle XY_0X'$ is

$$\mathcal{A}(\triangle XY_0X') = \frac{1}{2}t\cdot\overline{XY_0} = \left(\frac{1}{2}\cdot\frac{x}{2}\cdot\frac{c}{a}\right)\left(x\frac{b\cdot c}{b^2-a^2}\right) = \frac{x^2}{2^2}\cdot\frac{b}{a}\cdot\frac{c^2}{b^2-a^2} \quad (12)$$

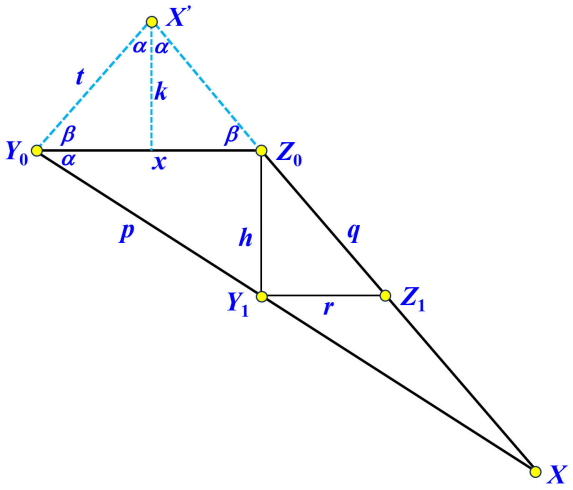


Figure 9: Jackson and Johnson's Proof: Part 2

The area of $\triangle XY_0X'$ may also be calculated as the sum of the areas of $\triangle X'Y_0Z_0$ and $\triangle XY_0Z_0$:

$$\mathcal{A}(\triangle XY_0X') = \mathcal{A}(\triangle XY_0Z_0) + \mathcal{A}(\triangle X'Y_0Z_0) = \frac{x^2}{2^2}\left(\frac{b}{a}\right) + \frac{x^2}{2}\cdot\frac{ab}{b^2-a^2} = \frac{x^2}{2}\cdot\frac{b}{2a}\cdot\frac{a^2+b^2}{b^2-a^2} \quad (13)$$

Because the areas computed by Eqn. (12) and Eqn. (13) are the same, we have

$$\frac{x^2}{2^2}\cdot\frac{b}{a}\cdot\frac{c^2}{b^2-a^2} = \frac{x^2}{2}\cdot\frac{b}{2a}\cdot\frac{a^2+b^2}{b^2-a^2}$$

After a simple simplification, we have the desired result is $c^2 = a^2 + b^2$.

If $\alpha = \beta = 45°$, the altitude on the hypotenuse is $c/2$. The area of the right triangle can be computed in two ways: $a^2/2$ and $((c/2)\cdot c)/2$. Therefore, $a^2/2 = ((c/2)\cdot c)/2$ implies $a^2 + a^2 = c^2$.

4 Jackson and Johnson's Original Proof

Proof 9

The original proof of Jackson and Johnson used trigonometry based on side lengths $\overline{XY_0}$ and $\overline{XX'}$ and the law of sines that is independent of the Pythagorean Theorem and the Pythagorean Identity [7, 9]. We know the following from the previous section:

$$\overline{XY_0} = x\cdot\frac{bc}{b^2-a^2} \quad \text{and} \quad t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

To compute $\sin(2\alpha)$, we need $\overline{XX'} = t + \overline{XZ_0}$. Our technique gives $\overline{XZ_0}$ as follows:

$$\overline{XZ_0} = \frac{1}{1-\rho} \cdot q = x \cdot \frac{ac}{b^2 - a^2}$$

Therefore, we have

$$\overline{XX'} = t + \overline{XZ_0} = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right) + x \cdot \frac{ac}{b^2 - a^2} = (x \cdot c)\frac{a^2 + b^2}{2a(b^2 - a^2)}$$

From the right triangle $\triangle XY_0X'$, with the help of Ean (10) we have $\sin(2\alpha)$ as follows:

$$\sin(2\alpha) = \frac{\overline{XY_0}}{\overline{XX'}} = \frac{x \cdot \frac{bc}{b^2-a^2}}{(x \cdot c)\frac{a^2+b^2}{2a(b^2-a^2)}} = \frac{2ab}{a^2 + b^2} \tag{14}$$

From the given triangle we have $\sin(\beta) = b/c$. From $\triangle X'Y_0Z_0$, the law of sines gives

$$\frac{\sin(2\alpha)}{x} = \frac{\sin(\beta)}{t} = \frac{(b/c)}{(x/2) \cdot (c/a)} = \frac{2ab}{x \cdot c^2}$$

Hence, we have the second way of computing $\sin(2\alpha)$:

$$\sin(2\alpha) = \frac{2ab}{c^2} \tag{15}$$

The results from Eqn (14) and Eqn (15) must be equal. Then, it is obvious that the desired result $a^2 + b^2 = c^2$ holds. Because we know $\sin(2\alpha) = 2ab/(a^2 + b^2)$, the above discussion yields the Pythagorean Theorem and the double angle formula for $\sin(x)$ at the same time. Obviously, the only difference between Jackson and Johnson's original proof and the proof in the previous section is the use of length and trigonometry vs. the use of area.

5 Possibly New Proofs

This section presents our (possibly) new proofs based on the concept and properties of the Lemoine point. Section 5.1 introduces the concepts of symmedian and the Lemoine point and some properties. Then, Section 5.2 presents three more proofs.

5.1 Symmedians and the Lemoine Point of a Triangle

A vertex of a triangle has an *angle bisector* that bisects the angle of that vertex and a *median* that is the line joining that vertex and the midpoint of that vertex's opposite side. In Figure 10, the black solid line is the angle bisector of angle C, the red dashed line is the median, and the blue dashed line is the line symmetric to the median with respect to the angle bisector. This line is referred to as the *symmedian* of the median at that vertex. Note that the angle between the angle bisector and

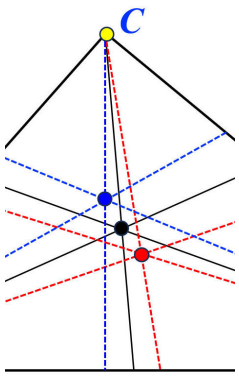


Figure 10: The Symmedian at a Vertex

the median is equal to the angle between the angle bisector and the symmedian. Or, the bisector bisects the angle between the median and the symmedian.

Each triangle has three vertices and hence three symmedians. It is known that these three symmedians are concurrent. The point where the three symmedian lines meet is referred to as the *Lemoine* point, the *Grebe* point or the *Symmedian* point (Figure 11). In this note we shall use "Lemoine point"exclusively. This point plays a significant role in modern triangle geometry.

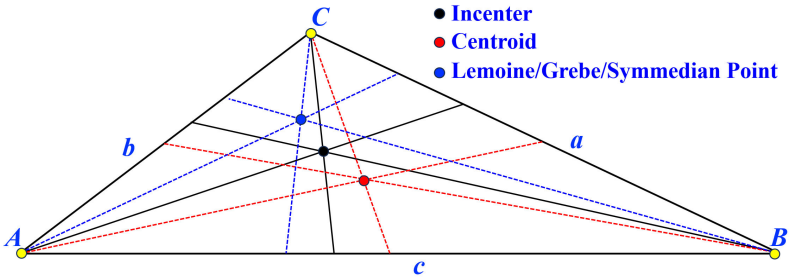


Figure 11: The Lemoine/Grebe/Symmedian Point

Suppose a triangle $\triangle ABC$ has all three squares on its sides (Figure 12). Extending the outer side of each square creates a similar right triangle $\triangle A_0B_0C_0$. It is clear that $\triangle ABC \sim \triangle A_0B_0C_0$. Because the corresponding sides are parallel, their intersection points are collinear (*i.e.*, meeting at the line at infinity) and by Desargues' Theorem the lines joining the corresponding vertices are concurrent at a point K. This point K is exactly the *Lemoine* point of $\triangle ABC$ and $\triangle A_0B_0C_0$ (Gallatly [2, Chap X, p.86]).

In fact, K is the *homothetic* center of $\triangle ABC$ and $\triangle A_0B_0C_0$. It is also known as the *center of similarity* or *center of similitude* of $\triangle ABC$ and $\triangle A_0B_0C_0$. Let the distances from K to sides a, b and c be p_a, p_b and p_c, respectively. An important property of K is $p_a : p_b : p_c = a : b : c$ or $p_a/a = p_b/b = p_c/c$ (Honsberger [3, p. 59]). This property can be used as a characterization of

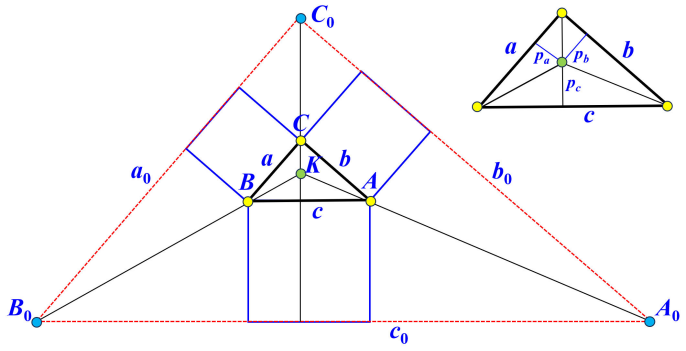


Figure 12: The Lemoine Point Construction

the Lemoine/Grebe/Symmedian point.

For a right triangle, the Lemoine point is the midpoint of the altitude on the hypotenuse. Suppose $\triangle ABC$ is a right triangle with $\angle C = 90°$. We need to show that (1) the altitude on the hypotenuse is an symmedian and (2) the midpoint of this symmedian is the Lemoine point.

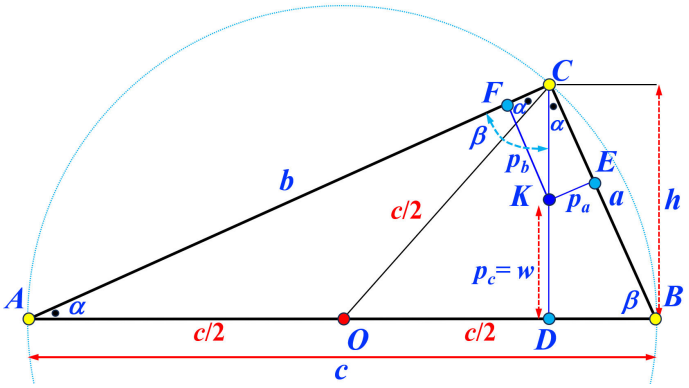


Figure 13: The Lemoine/Grebe/Symmedian Point of a Right Triangle

The first thing we need to show is that *the altitude* $\overleftrightarrow{CD}$ *is actually a symmedian*. This is not difficult to do. Suppose $\angle A$ and $\angle B$ of $\triangle ABC$ be α and β (Figure 13). The line joining the midpoint O of side $\overleftrightarrow{AB}$ and C is a median. Because $\triangle ABC$ is a right triangle, O is the center of its circumcircle whose radius is $c/2$. Hence, $\triangle OAC$ is an isosceles with $\angle OAC = \angle OCA = \alpha$. Let the altitude on the hypotenuse be $\overleftrightarrow{CD}$. Because $\angle BCD = \alpha$, the altitude is symmetric to the median $\overleftrightarrow{CO}$ with respect to the angle bisector of $\angle C$. As a result, the altitude on the hypotenuse is an symmedian.

Now we need to show the Lemoine point is the midpoint of $\overline{CD}$. Let K be a point on the altitude and the distances from K to side a, b and c be p_a, p_b and p_c. For convenience, let $p_c = w = r \cdot h$, where h is the length of the altitude and $0 < r < 1$. In this way, p_c is measured based on the ratio

related to h. Because $\triangle ABC \sim \triangle CKE$, we have

$$\frac{p_a}{a} = \frac{\overline{CK}}{c} = \frac{h(1-r)}{c}$$

Because $\triangle ABC \sim \triangle KCF$, we have

$$\frac{p_b}{b} = \frac{h(1-r)}{c}$$

If K is the Lemoine point, we must have

$$\frac{p_a}{a} = \frac{p_b}{b} = \frac{p_c}{c} = \frac{r \cdot h}{c}$$

This implies

$$\frac{h(1-r)}{c} = \frac{h \cdot r}{c}$$

Therefore, if K is the Lemoine point, $r = 1/2$ and the Lemoine point is the midpoint of the altitude on the hypotenuse.

5.2 New Proofs Based on the Lemoine Point

Proof 10

We showed that K is the midpoint of the altitude $\overline{CD}$. Hence, $p_c = h/2$. Because $h = (a \cdot b)/c$, we have $p_c = h/2 = ((a \cdot b)/c)/2 = (a \cdot b)/(2c)$. Because of $p_a/a = p_b/b = p_c/c = (a \cdot b)/(2c^2)$, we have p_a, p_b and p_c as follows:

$$p_a = \frac{a^2 b}{2c^2}, \quad p_b = \frac{ab^2}{2c^2} \quad \text{and} \quad p_c = \frac{ab}{2c} \tag{16}$$

Using the areas of the three trapezoids $\mathcal{A}(AA_0B_0B)$, $\mathcal{A}(BB_0C_0C)$ and $\mathcal{A}(CC_0A_0A)$ the area of $\triangle A_0B_0C_0$ is calculated easily with our method. Note that p_c and the $p_c + c$ are the lengths of the altitude of $\triangle ABC$ and $\triangle A_0B_0C_0$ on the hypotenuse. Because $\triangle ABC \sim \triangle A_0B_0C_0$, $\rho = c/c_0 = p_c/(p_c + c)$ and hence the scaling factor ρ going from $\triangle A_0B_0C_0$ to $\triangle ABC$ is

$$\rho = \frac{p_c}{p_c + c} = \frac{\frac{ab}{2c}}{\frac{ab}{2c} + c} = \frac{a \cdot b}{a \cdot b + 2c^2} \quad \text{and} \quad \frac{1-\rho}{\rho} = \frac{2c^2}{ab} \tag{17}$$

Because $a_0 = a/\rho$, $b_0 = b/\rho$ and $c_0 = c/\rho$, the areas of trapezoids AA_0C_0C, BC_0B_0B and AA_0B_0B are as follows:

$$\begin{aligned}
\mathcal{A}(AA_0C_0C) &= \frac{1}{2}(a + a_0) \cdot a = \frac{1}{2}\left(a + \frac{a}{\rho}\right) \cdot a = \frac{a^2}{2}\left(1 + \frac{1}{\rho}\right) \\
\mathcal{A}(BC_0B_0B) &= \frac{1}{2}(b + b_0) \cdot b = \frac{1}{2}\left(b + \frac{b}{\rho}\right) \cdot b = \frac{b^2}{2}\left(1 + \frac{1}{\rho}\right) \\
\mathcal{A}(AA_0B_0B) &= \frac{1}{2}(c + c_0) \cdot c = \frac{1}{2}\left(c + \frac{c}{\rho}\right) \cdot c = \frac{c^2}{2}\left(1 + \frac{1}{\rho}\right)
\end{aligned}$$

The area sum of all three trapezoids is

$$\mathcal{A}(\text{outer ring of } \triangle ABC) = \left(1+\frac{1}{\rho}\right)\frac{a^2+b^2+c^2}{2}$$

Therefore, the area of $\triangle A_0B_0C_0$ according to our method is

$$\mathcal{A}(A_0B_0C_0) = \frac{1}{1-\rho^2}\left[\frac{a^2+b^2+c^2}{2}\left(1+\frac{1}{\rho}\right)\right] = \frac{1}{2}\frac{a^2+b^2+c^2}{\rho(1-\rho)} \tag{18}$$

The area of $\triangle A_0B_0C_0$ may also be computed as follows:

$$\mathcal{A}(\triangle A_0B_0C_0) = \frac{1}{2}\overline{A_0C_0}\cdot\overline{B_0C_0} = \frac{1}{2}\left(\frac{a}{\rho}\right)\cdot\left(\frac{b}{\rho}\right) = \frac{1}{2}\cdot\frac{a\cdot b}{\rho^2}$$

This result must agree with the one shown in Eqn. (18) and we have the following:

$$\frac{1}{2}\frac{a^2+b^2+c^2}{\rho(1-\rho)} = \mathcal{A}(A_0B_0C_0) = \frac{1}{2}\cdot\frac{a\cdot b}{\rho^2}$$

Simplifying yields

$$\frac{a^2+b^2+c^2}{1-\rho} = \frac{a\cdot b}{\rho} \quad \text{or} \quad a^2+b^2+c^2 = \frac{1-\rho}{\rho}\cdot(a\cdot b)$$

and after plugging the value of ρ Eqn. (17) followed by a very simple simplification we have $c^2 = a^2+b^2$.

Proof 11

It is worthwhile to note that with the property of the Lemoine point for right triangle in hand, a simpler proof is possible. Recall the results in Eqn (16), the area of $\triangle ABC$ is the sum of three smaller triangles $\triangle KAB$, $\triangle KBC$ and $\triangle KCA$ (Figure 14):

$$\mathcal{A}(ABC) = \frac{1}{2}(p_a\cdot a + p_b\cdot b + p_c\cdot c) = \frac{a\cdot b}{2}\left(\frac{a^2}{2c^2}+\frac{b^2}{2c^2}+\frac{1}{2}\right) = \frac{a\cdot b}{2}\cdot\frac{a^2+b^2+c^2}{2c^2}$$

Because the above is equal to $(ab)/2$, after a simple simplification we have $c^2 = a^2+b^2$. This is a direct proof of the Pythagorean Theorem.

It is worthwhile to note that if K is the midpoint of the altitude on the hypotenuse, then p_a, p_b and p_c can be computed using similar triangles as in Eqn (16) and hence the same idea yields the Pythagorean Theorem. In this way, the concept of the Lemoine can be completely avoided.

Proof 12

Our next question is: *can K be selected as an arbitrarily point on the altitude*? Of course, K cannot be C and D. In this way, the distance p_c can be measured with respect to the length of the

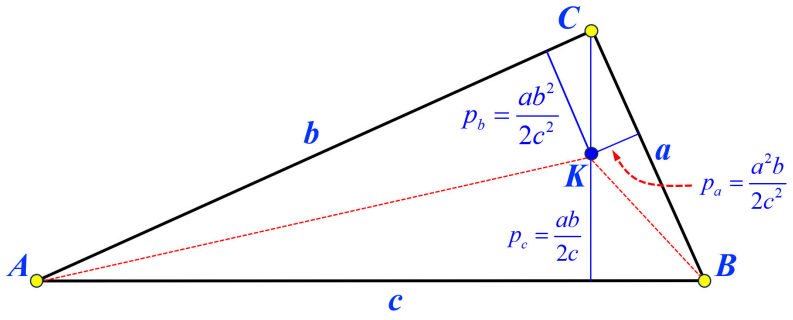


Figure 14: A Proof that Only Uses the Lemoine Point

altitude. Again, let h be the length of the altitude $\overline{CD}$ and $p_c = r \cdot h$ (Figure 13). We have already obtained the following at the beginning of this section:

$$\frac{p_a}{a} = \frac{p_b}{b} = \frac{(1-r)h}{c} \quad \text{and} \quad \frac{p_c}{c} = \frac{r \cdot h}{c}$$

and hence

$$p_a = \left(\frac{a}{c}\right) h(1-r), \quad p_b = \left(\frac{b}{c}\right) h(1-r) \quad \text{and} \quad p_c = h \cdot r$$

The area of $\triangle ABC$ is the sum of the areas of three triangles:

$$\begin{aligned}
\mathcal{A}(\triangle ABC) &= \mathcal{A}(\triangle KBC) + \mathcal{A}(\triangle KCA) + \mathcal{A}(\triangle KAB) \\
&= \frac{1}{2}a\left(\frac{a \cdot h}{c}(1-r)\right) + \frac{1}{2}b\left(\frac{b \cdot h}{c}(1-r)\right) + \frac{1}{2}c \cdot h \cdot r \\
&= \frac{1}{2}\left[\frac{a^2h}{c} - \frac{a^2h}{c}r + \frac{b^2h}{c} - \frac{a^2h}{c}r + c \cdot h \cdot r\right] \\
&= \frac{1}{2}\left[\frac{h}{c}\left(a^2+b^2\right) - h \cdot r\left(\frac{a^2}{c} + \frac{b^2}{c} - c\right)\right] \\
&= \frac{1}{2}\left(\frac{h}{c}\right)\left[\left(a^2+b^2\right) - r\left(a^2+b^2-c^2\right)\right]
\end{aligned}$$

Because the area of $\triangle ABC$ is also computed as $(h \cdot c)/2$, we have

$$\frac{1}{2}\left(\frac{h}{c}\right)\left[\left(a^2+b^2\right) - r\left(a^2+b^2-c^2\right)\right] = \frac{1}{2}(c \cdot h)$$

A simple simplification yields:

$$(a^2+b^2) - r(a^2+b^2-c^2) = c^2$$

Hence, we have

$$(1-r)\left(a^2+b^2-c^2\right)=0$$

Because $0<r<1$, $a^2+b^2-c^2=0$ and hence the Pythagorean Theorem holds.

Note that this proof still works when we set $r=0$. In this way, the values for p_a and p_b are still correct and $p_c=0$.

Proof 13

The previous proof actually provides another proof without the use of Lemoine point. In Figure 15 we have $K=D$ and $p_c=0$. Because $\triangle CDE\sim\triangle ABC\sim\triangle CDF$, we have

$$\frac{p_a}{h}=\frac{a}{c}\quad\text{and}\quad\frac{p_b}{h}=\frac{b}{c}$$

We know that $h=(a\cdot b)/c$. Plugging h into p_a and p_b yields:

$$\begin{aligned}p_a &= h\cdot\frac{a}{c}=\left(\frac{a\cdot b}{c}\right)\left(\frac{a}{c}\right)=\frac{a^2b}{c^2}\\ p_b &= h\cdot\frac{b}{c}=\left(\frac{a\cdot b}{c}\right)\left(\frac{b}{c}\right)=\frac{a\cdot b^2}{c^2}\end{aligned}$$

The area of $\triangle ABC$ is the sum of the areas of the rectangle $CEDF$ and the two right triangles $\triangle DBE$ and $\triangle ADF$. The area of rectangle $CEDF$ is

$$\mathcal{A}(CEDF)=p_a\cdot p_b=\frac{a^2b}{c^2}\cdot\frac{a\cdot b^2}{c^2}=\frac{a^3b^3}{c^4}\tag{19}$$

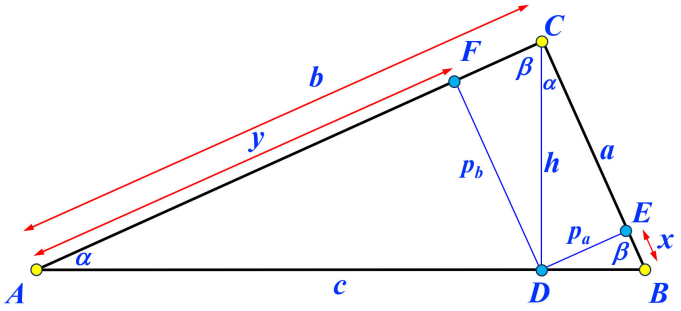


Figure 15: A Special Case for a Right Triangle

Let x and y be the length of segments EB and FA, respectively. Because $\triangle BDE\sim\triangle BAC$, we have $x/p_a=a/b$. Because $\triangle ADF\sim\triangle ABC$, we have $y/p_b=b/a$. Hence, x and y in terms of a, b

and c are

$$\begin{aligned} x &= p_a \cdot \frac{a}{b} = \frac{a^2 b}{c^2} \cdot h = \frac{a^2 b}{c^2} \cdot \frac{a \cdot b}{c} = \frac{a^3}{c^2} \\ y &= p_b \cdot \frac{b}{a} = \frac{ab^2}{c^2} \cdot h = \frac{ab^2}{c^2} \cdot \frac{a \cdot b}{c} = \frac{b^3}{c^2} \end{aligned}$$

The areas on $\triangle DBE$ and $\triangle ADF$ are computed as follows:

$$\begin{aligned} \mathcal{A}(\triangle DBE) &= \frac{1}{2} x \cdot p_a = \frac{1}{2} \cdot \frac{a^3}{c^2} \cdot \frac{a^2 b}{c^2} = \frac{1}{2} \frac{a^5 b}{c^4} \\ \mathcal{A}(\triangle DAF) &= \frac{1}{2} y \cdot p_b = \frac{1}{2} \cdot \frac{b^3}{c^2} \cdot \frac{ab^2}{c^2} = \frac{1}{2} \frac{ab^5}{c^4} \end{aligned}$$

The area of $\triangle ABC$ is calculated as follows:

$$\begin{aligned} \mathcal{A}(\triangle ABC) &= \mathcal{A}(CEDF) + \mathcal{A}(\triangle DBE) + \mathcal{A}(\triangle ADF) \\ &= \frac{a^3 b^3}{c^4} + \frac{1}{2} \frac{a^5 b}{c^4} + \frac{1}{2} \frac{ab^5}{c^4} \\ &= \frac{a \cdot b}{c^4} \cdot \frac{a^4 + b^4 + 2a^2 b^2}{2} \\ &= \frac{a \cdot b}{c^4} \cdot \frac{(a^2 + b^2)^2}{2} \end{aligned}$$

This area computation must agree with the known one $(a \cdot b)/2$:

$$\frac{a \cdot b}{c^4} \cdot \frac{(a^2 + b^2)^2}{2} = \frac{1}{2}(a \cdot b)$$

Then we have $(a^2 + b^2)^2 = c^4$. However, this is equivalent to $(a^2 + b^2)^2 - (c^2)^2 = 0$ and hence we have

$$(a^2 + b^2 - c^2)(a^2 + b^2 + c^2) = 0$$

Because $a^2 + b^2 + c^2$ cannot be 0, we must have $a^2 + b^2 - c^2 = 0$ and the Pythagorean Theorem follows.

6 Conclusions

We developed an easy and effective way for proving the Pythagorean Theorem. This method is based on a simple principle of similarity. Given a shape A and a similar shape $\mathsf{B} \subseteq \mathsf{A}$, if the scaling factor from A to B is ρ $(0 < \rho < 1)$, then the area of A is computed as $\mathcal{A}(\mathsf{A}) = \mathcal{A}(\mathsf{A} - \mathsf{B})/(1 - \rho^2)$. Note that even though this method is only applied to right triangles, it can be used with general shapes. This method is applied to several classical proofs in Lommis [6] and to new proofs. In particular, the use of trigonometry in Jackson and Johnson's proof [4] is eliminated

becoming a geometrical one. With the help of the Lemoine Point, we have a short and simple direct proof and another one based on our method. The Appendix includes Zimba's proof of the angle difference identities of sin() and cos() being independent of the Pythagorean Theorem. We also show that a similar technique can be used to prove the angle sum identities being independent of the Pythagorean Theorem. Then, the computation of the derivatives of sin() and cos() is derived from the angle sum identities, and, finally, with the help of L'Hôpital's Rule the double angle identities are used to proof $\sin^2(x)+\cos^2(x)=1$. Consequently, this manuscript successfully demonstrated that many fundamental formulae of trigonometry are independent of the Pythagorean Theorem and the $\sin^2(x)+\cos^2(x)=1$ identity.

A Proofs of Some Important Trigonometric Identities

We present Zimba's proof in [13] that the angle difference identities for sin() and cos() are independent of the Pythagorean Theorem and the Pythagorean Identity. With a similar technique, we prove that the angle sum identities for sin() and cos() are also independent of the Pythagorean Theorem. Furthermore, the derivative computation of sin() and cos() share the same property and the double angle identities can be used to prove the Pythagorean Identity. This firmly shows that Loomis's claim is false and that even though analytic geometry uses the Cartesian Coordinate System many fundamental results are independent of the Pythagorean Theorem and the Pythagorean Identity.

A.1 Zimba's Proof of the Angle Difference Identities

Without loss of generality, we assume $0<\beta\leq\alpha<90^\circ$ in this section because the main focus is a right triangle. Consider Figure 16. Line $\overleftrightarrow{OQ}$ makes an angle of $\alpha-\beta$ with the x-axis, where $\overline{OQ}=1$. Let line $\overleftrightarrow{OP}$ make an angle of β with $\overleftrightarrow{OQ}$, where P is the perpendicular foot from Q to $\overleftrightarrow{OP}$. Thus, $\overleftrightarrow{OP}$ makes an angle of α with the x-axis. From P and Q drop perpendiculars to the x-axis meeting it at S and T. Therefore, we have $\overline{QT}=\sin(\alpha-\beta)$ and $\overline{OT}=\cos(\alpha-\beta)$. From $\triangle OPQ$ we have $\overline{PQ}=\sin(\beta)$ and $\overline{OP}=\cos(\beta)$.

In $\triangle OPS$, because $\sin(\alpha)=\overline{PS}/\overline{PO}=\overline{PS}/\cos(\beta)$ we have $\overline{PS}=\sin(\alpha)\cos(\beta)$. Similarly, we have $\overline{OS}=\cos(\alpha)\cos(\beta)$. From Q drop a perpendicular to $\overleftrightarrow{PS}$ meeting it at R. Note that $\angle P$ of $\triangle PQR$ is α. In $\triangle PQR$, because $\sin(\alpha)=\overline{QR}/\overline{QP}=\overline{QR}/\sin(\beta)$ we have $\overline{QR}=\sin(\alpha)\sin(\beta)$. Similarly, we have $\overline{PR}=\cos(\alpha)\sin(\beta)$. Consequently, the desired results are as follows:

$$\begin{aligned}\sin(\alpha-\beta) &= \overline{QT}=\overline{PS}-\overline{PR}=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ \cos(\alpha-\beta) &= \overline{OS}+\overline{ST}=\overline{OS}+\overline{RQ}=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\end{aligned}$$

If $\alpha=\beta$, we have the following:

$$1=\cos(0)=\cos(\alpha-\alpha)=\cos^2(\alpha)+\sin^2(\alpha)$$

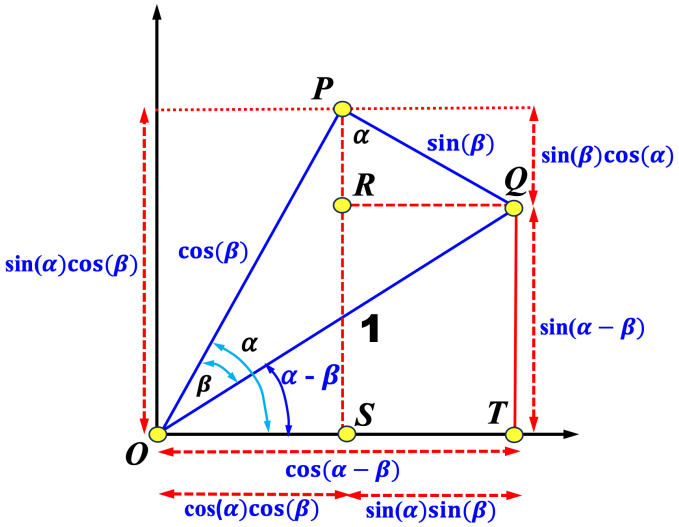


Figure 16: Proof of the Angle Difference Identities

A.2 The Angle Sum Identities

We shall prove the angle sum identities for sin() and cos() based on Zimba's approach. From O construct a line $\overleftrightarrow{OP}$ that makes an angle of $\alpha+\beta$ with the x-axis and $\overline{OP}=1$ (Figure 17). From O construct a line $\overleftrightarrow{OQ}$ that makes an angle α with the x-axis such that Q is the perpendicular foot from P to $\overleftrightarrow{OQ}$. In this way, the angle between $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$ is β. Let the perpendicular feet from P and Q to the x-axis be S and T. From Q construct a perpendicular to $\overleftrightarrow{PS}$ meeting it at R. Hence, we have $\sin(\alpha+\beta)=\overline{PS}$, $\cos(\alpha+\beta)=\overline{OS}$, $\sin(\beta)=\overline{PQ}$ and $\cos(\beta)=\overline{OQ}$.

From $\triangle OQT$, because $\sin(\alpha)=\overline{QT}/\overline{QO}=\overline{QT}/\cos(\beta)$ we have $\overline{QT}=\sin(\alpha)\cos(\beta)$. Similarly, we have $\overline{OT}=\cos(\alpha)\cos(\beta)$. From $\triangle PQR$, because $\sin(\alpha)=\overline{QR}/\overline{QP}=\overline{QR}/\sin(\beta)$ we have $\overline{QR}=\sin(\alpha)\sin(\beta)$. Similarly, we have $\overline{PR}=\cos(\alpha)\sin(\beta)$. Therefore, we have:

$$\begin{aligned}\sin(\alpha+\beta) &= \overline{PR}+\overline{RS}=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ \cos(\alpha+\beta) &= \overline{OT}-\overline{ST}=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\end{aligned}$$

Consequently, the angle sum identities for sin() and cos() are independent of the Pythagorean Theorem and the Pythagorean Identity.

A.3 $\frac{d\sin(x)}{dx}$ and $\frac{d\sin(x)}{dx}$ Are Independent of the Pythagorean Theorem

The angle sum and angle difference identities give the following:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\end{aligned}$$

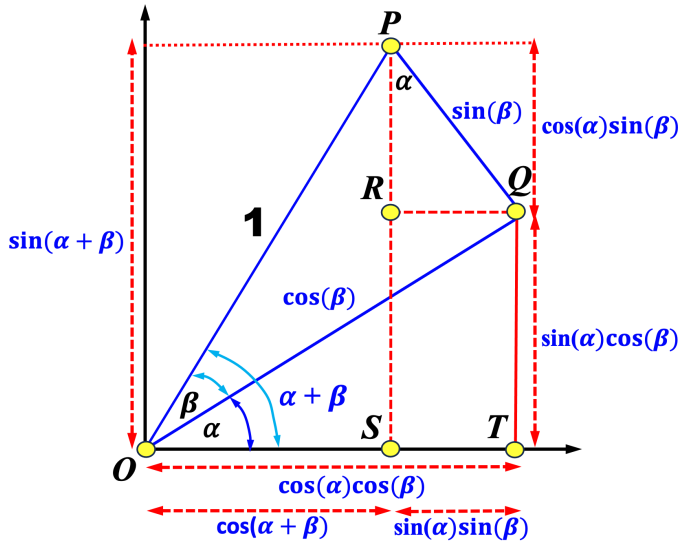


Figure 17: Proof of the Angle Sum Identities

Subtracting the second from the first yields

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos(\alpha)\sin(\beta)$$

Let $p=\alpha+\beta$ and $q=\alpha-\beta$. Then, $\alpha=(p+q)/2$ and $\beta=(p-q)/2$. Plugging p and q into the above identity gives one of the *sum-to-product* identities:

$$\sin(p)-\sin(q)=2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$$

Then, the derivative of $\sin()$ is computed as follows:

$$\begin{aligned}\frac{d\sin(x)}{dx} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\ &= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\\ &= \left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\cdot\left[\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\right]\\ &= \cos(x)\end{aligned}$$

As $h\to 0$, the first term approaches $\cos(x)$ while the second approaches 1. Note that $\lim_{h\to 0}\sin(h)/h=1$ does not depend on the Pythagorean Theorem. Because $\cos(x)=\sin(\pi/2-x)$, by the Chain Rule we have $\frac{d\cos(x)}{dx}=\frac{d\sin(\pi/2-x)}{dx}=\cos(\pi/2-x)\frac{d(\pi/2-x)}{dx}=-\cos(\pi/2-x)=-\sin(x)$ and the calculation of $\frac{d\sin(x)}{dx}$ and $\frac{d\sin(x)}{dx}$ is independent of the Pythagorean Theorem and the Pythagorean Identity.

A.4 The Double Angle Identities

We have shown that $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$ and the derivatives of $\sin()$ and $\cos()$ are all independent of the Pythagorean Theorem and the Pythagorean Identity. We now prove $\sin^2(x) + \cos^2(x) = 1$ with the double angle identities:

$$\begin{aligned} \sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\ \cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \end{aligned}$$

Because of the following:

$$\begin{aligned} \sin(x) &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\ \cos(x) &= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \end{aligned}$$

we have

$$\sin^2(x) + \cos^2(x) = \left(\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)\right)^2$$

With the same technique, we have:

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= \left(\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)\right)^2 \\ &= \left(\left(\sin^2\left(\frac{x}{4}\right) + \cos^2\left(\frac{x}{4}\right)\right)^2\right)^2 \\ &= \left(\sin^2\left(\frac{x}{2^2}\right) + \cos^2\left(\frac{x}{2^2}\right)\right)^{2^2} \\ &\vdots \\ &= \left(\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right)^{2^n} \end{aligned}$$

We need to prove the following:

$$\lim_{n\to\infty}\left[\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = 1$$

The left-hand side of the above can be rewritten as

$$\begin{aligned} \left[\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} &= \exp\left(2^n \ln\left(\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right)\right) \\ &= \exp\left(\frac{\ln\left(\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right)}{\frac{1}{2^n}}\right) \end{aligned}$$

For convenience, let $h = 1/2^n$. Therefore, as $n \to \infty$, $h \to 0$ and the above becomes

$$\left[\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = \exp\left(\frac{\ln(\sin^2(xh) + \cos^2(xh))}{h}\right)$$

As $h \to 0$, the numerator approaches $\ln(\sin^2(0)+\cos^2(0)) = \ln(1) = 0$ and the denominator approaches 0. As a result, we have an indefinite form of $0/0$ and L'Hôpital's Rule is needed to compute the limit. The derivative of $\ln(\sin^2(xh)+\cos^2(xh))$ with respect to h is

$$\begin{aligned}
\frac{d(\ln(\sin^2(xh)+\cos^2(xh)))}{dh} &= \frac{1}{\sin^2(xh)+\cos^2(xh)} \frac{d(\sin^2(xh)+\cos^2(xh))}{dh} \\
&= \frac{1}{\sin^2(xh)+\cos^2(xh)} (2\sin(xh)\cos(xh)x + 2\cos(xh)(-\sin(xh))x) \\
&= 0
\end{aligned}$$

The derivative of the denominator is 1. As a result, we have

$$\lim_{n\to\infty} \left[\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = \exp(0) = 1$$

Consequently, $\sin^2(x)+\cos^2(x) = 1$ holds.

A.5 A Simple Calculus Based Proof

Finally, we offer a very simple proof based on calculus. Let function $f(x)$ be defined as follows:

$$f(x) = \cos^2(x) + \sin^2(x)$$

Differentiate this function yields:

$$\frac{df(x)}{dx} = \frac{d(\cos^2(x)+\sin^2(x))}{dx} = 2\cos(x)(-\sin(x)) + 2\sin(x)\cos(x) = 0$$

Therefore, $f(x)$ is a constant function for some c:

$$f(x) = \sin^2(x) + \cos^2(x) = c$$

Because $\sin(0) = 0$ and $\cos(0) = 1$, we have

$$f(x) = \sin^2(x) + \cos^2(x) = 1$$

This proves the Pythagorean Identity. Note that the computation of the derivatives of $\sin(x)$ and $\cos(x)$ is independent of the Pythagorean Theorem and the Pythagorean Identity. Consequently, the above proof is valid.

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1. First Draft: August 21, 2023

2. Typos Corrected and Abstract and Appendix Added: September 19, 2023

3. Typos/Diagrams Corrected + Images & New Material Added: November 3, 2023

4. Partially Rewritten: January 15, 2024.

5. A Minor Typo in Section A.5 Corrected: February 28, 2024.

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