

path length and vulnerability for "ideal" networks against which network designers may measure their networks. For this reason, we too, determine the lower bounds on these performance measures for regular graphs. In addition, the analysis provides a uniform basis by which to compare the various algorithms.

Section 4.2 proposes a set of performance measures, and section 4.3 reviews and extends some properties of regular graphs. Sections 4.4 to 4.8 determine the performance of the various broadcast routing algorithms. Section 4.9 compares the various algorithms, and 4.10 determines average values of the performance measures for the ARPANET topology.

## 4.2 Performance Measures

This section develops some performance measures for broadcast communication in store-and-forward networks. We develop the performance measures by considering the overhead imposed on the communication subnet by a broadcast, and the delays in performing the broadcast.

An important measure of the amount of bandwidth used in performing a broadcast, and of the processing overhead in the switching (store-and-forward) nodes of the subnet is the Number of Packets Transmitted,  $NPT(i)$ , in broadcasting a message initiated from node  $i$  in the subnet.  $NPT(i)$  is in units of packet-hops per broadcast.

The Broadcast Delay,  $BD(i,j)$ , is the delay before node  $j$  receives a broadcast message initiated from  $i$ .  $BD(i,j)$  is in appropriate units for delay, e.g. seconds or hops. The average Broadcast Delay,  $BDav(i)$ , from a node  $i$  that initiates the broadcast is an estimate of the delay before a receiver hears the message.  $BDav(i)$  is therefore a measure of the ability of the broadcast routing scheme to deliver messages quickly when initiated from  $i$ . In a communication subnet with  $N$  nodes

$$BDav(i) = \frac{1}{N-1} * \sum_{\substack{j=1 \\ j \neq i}}^N BD(i,j). \quad (4.1)$$

Analogously,  $BDmax(i)$  is the maximum Broadcast Delay from a broadcaster  $i$ , and therefore is an indication of the maximum delay before all receivers have heard the message.  $BDmax(i)$  is, therefore, also a measure of the cost of broadcast from node  $i$ .



$$BD_{\max}(i) = \max \{BD(1,j) \mid 1 \leq j \leq N, j \neq 1\}. \quad (4.2)$$

These measures are useful in the design of timeouts for the reinitiation of the broadcast, or recovery techniques in distributed operating systems that use the broadcast routing feature of the communication subnet to locate resources.

The Broadcast Cost, BC, in a communication network is the sum of the cost of broadcast with each node as the initiator. It is assumed that each node has equal probability of initiating a broadcast.  $BD_{\max}(i)$  is the cost of broadcast when initiated from node  $i$ . Hence, if there are  $N$  nodes in the network, then

$$BC = \sum_{i=1}^N BD_{\max}(i). \quad (4.3)$$

These measures of performance can easily be determined for a particular network using a particular broadcast routing technique. The performance measures are a function of the topology of the network. In order to determine lower bounds on these measures, we examine regular graphs.

### 4.3 Regular Graphs

In this section we review and extend some of the definitions and properties of regular graphs, Moore graphs, and generalized Moore graphs.

We assume that the number of nodes in the graph is  $N$ , and that all edge costs are the same and equal to one. The degree of a node in a graph is the number of edges incident to that node. If the degree of all the nodes is same and equal to  $D$ , then the graph is called regular with degree  $D$ . Figure 4.1 shows some regular graphs. A theorem and two corollaries may be stated concerning the relationship between  $N$  and  $D$  in regular graphs. These are simple, and stated without proof.

Theorem 4.1: The sum of the degrees of the nodes of any graph is twice the number of edges in the graph.

Corollary 4.1.1: In any graph the number of nodes of odd degree is even.

Corollary 4.1.2: A regular graph with degree  $D$  and  $N$  nodes can always be constructed if  $N \cdot D$  is even.

Consider a regular graph with degree  $D$ . The maximum number of nodes at distance one from any node is  $D$ , at distance two  $D(D-1)$ , at distance three  $D(D-1)^2$  and so on. Moore graphs are those regular graphs that have exactly  $D(D-1)^{j-1}$  nodes at distance  $j$ , for any node considered. Figure 4.2a illustrates such a tree as seen from one node  $X$ , where the graph has 10 nodes and is of degree 3. Such a picture can



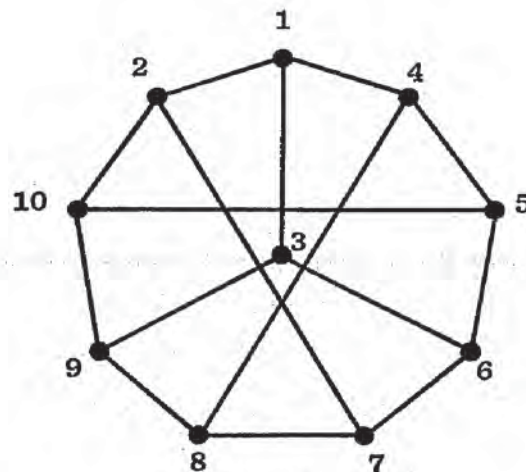
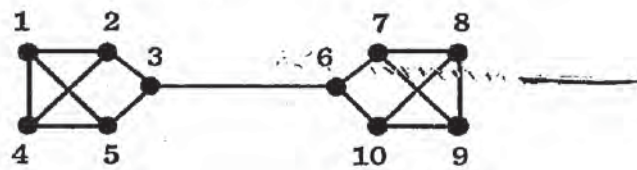
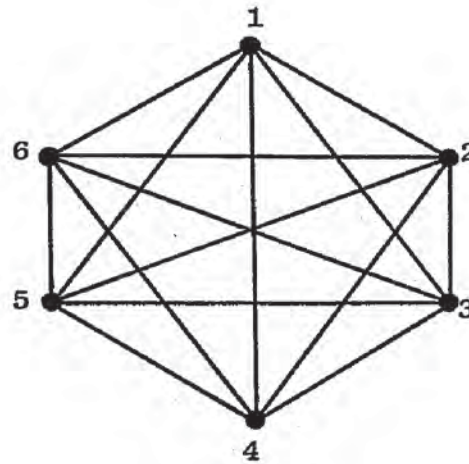


Figure 4.1. SOME REGULAR GRAPHS.

be drawn for all nodes in the graph. Thus one must join all nodes at the highest numbered level together in such a way to meet this constraint. This creates a sort of "spaghetti junction" at the highest level. Figure 4.2b illustrates how the tree of figure 4.2a can be converted into a Moore graph. Figure 4.2c is a more aesthetic representation of figure 4.2b. This is the Petersen Graph. Moore graphs satisfy the following relation:

$$N = 1 + D * \sum_{j=1}^m (D-1)^{j-1}. \quad (4.4)$$

where  $j$  is the level number and  $m$  the highest level. Conditions under which a given graph can be a Moore graph have been derived in [Hoffman60, Cerf73].

Of course, not all graphs have a number of nodes that satisfy equation (4.4). In this case the extra nodes are placed at the highest numbered level,  $m$ , and one must then join the nodes at level  $m$  and  $m-1$  such that the constraints are satisfied. Such graphs are called generalized Moore graphs and have been studied in detail by Cerf, Cowan, Mullin and Stanton [Cerf73, Cerf74d, Cerf74e]. A generalized Moore graph having 16 nodes and of degree 3 is illustrated in figure 4.3.

We now introduce a new term. A pseudo generalized Moore graph is a regular graph such that for at least one node, the number of nodes at distance  $j$  is  $D(D-1)^{j-1}$  for all  $j > 1$  except possibly the highest numbered level which may be incompletely filled. In other words, from the point of view of that node, the graph looks like a filled tree except for the



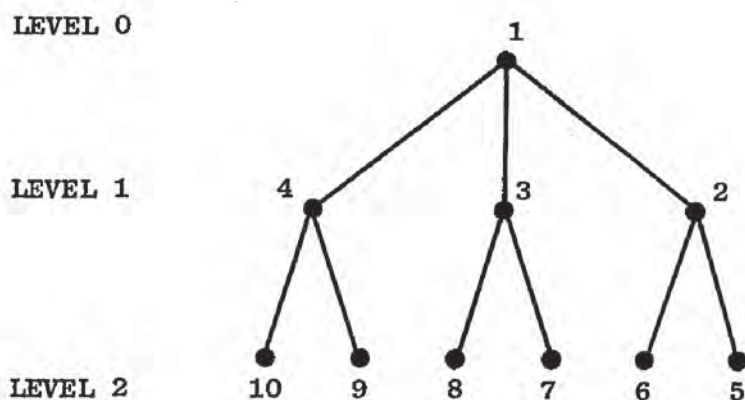


Figure 4.2a

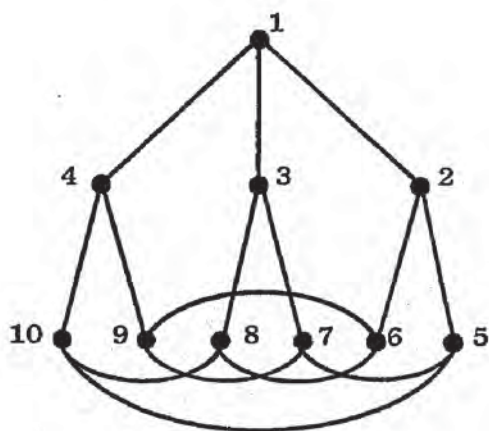


Figure 4.2b

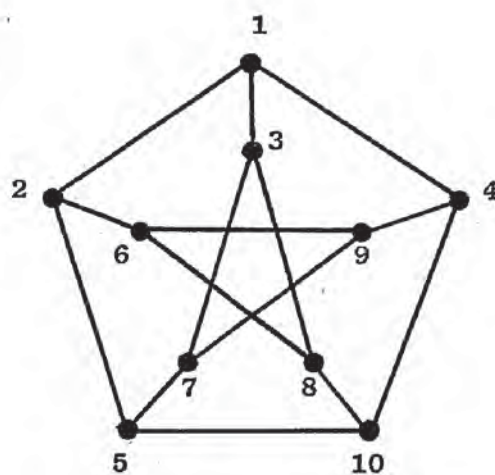


Figure 4.2c

Figure 4.2. A MOORE GRAPH.

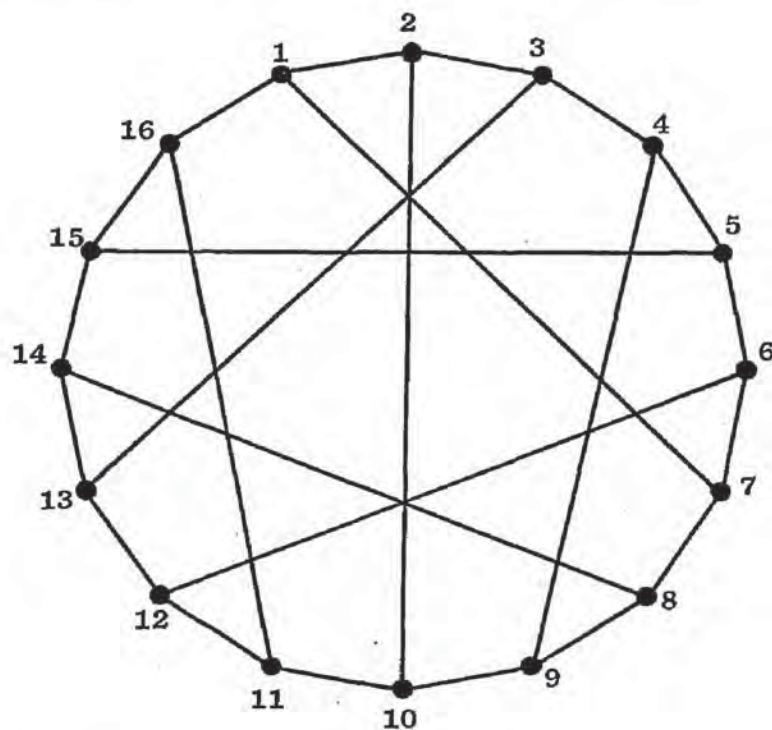


Figure 4.3. A GENERALIZED MOORE GRAPH HAVING 16 NODES AND DEGREE 3.

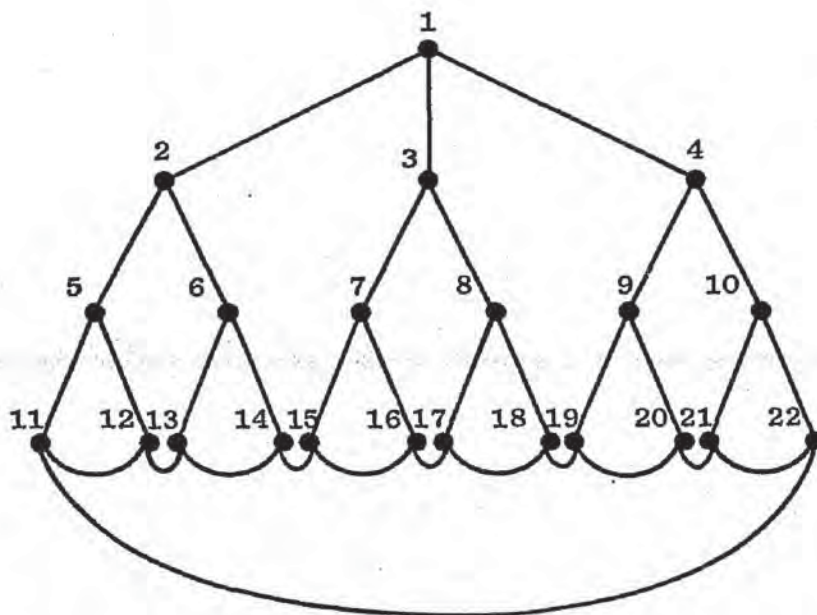


Figure 4.4. A PSEUDO GENERALIZED MOORE GRAPH HAVING 22 NODES AND OF DEGREE 3.



spaghetti junction at the last two levels. Figure 4.4 illustrates a pseudo generalized Moore graph of degree 3 having 22 nodes. A Moore graph does not exist for such a regular graph [Cerf73], even though its highest numbered level is completely filled, and so neither does a generalized Moore graph.

**Theorem 4.2:** For a given  $N$  and  $D$  such that a regular graph exists, a pseudo generalized Moore graph also exists.

**Proof:** The theorem is proved by showing that a pseudo generalized Moore graph can always be constructed if a regular graph can.

Take any node and connect it to  $D$  other nodes. Take each of these  $D$  nodes and connect each to  $D-1$  other nodes that have not already been connected to, so as to form a tree. Continue connecting the leaves of the tree to  $D-1$  other nodes until all  $N$  nodes have been used. The result is a minimum height tree with the starting node as the root. The highest level  $m$  may be incompletely filled, where  $m \geq 1$ .

We now show that if  $N \cdot D$  is even then the nodes at level  $m$  and  $m-1$  that do not have degree  $D$  can be connected to one another so as to produce a regular graph.

Let  $U$  be the number of nodes by which the  $m^{\text{th}}$  level is unfilled. If the  $m^{\text{th}}$  level was completely filled then it would have  $D(D-1)^{m-1}$  nodes that would give rise to  $D(D-1)^m$  branches. However, there are  $U$  less nodes at the  $m^{\text{th}}$  level and therefore the  $m^{\text{th}}$  level gives rise to  $U(D-1)$  less branches, but the  $(m-1)^{\text{th}}$  level

gives rise to an additional  $U$ . Therefore the total number of branches that must be connected to one another

$$= D(D-1)^m + U - U(D-1)$$

$$= D(D-1)^m - U(D-2).$$

If this number is always even, then the branches can be appropriately paired off producing a regular graph. The first term,  $D(D-1)^m$  is always even for any  $D$  since  $m \geq 1$ . When  $D$  is even, the second term is also even. We now show that when  $D$  is odd,  $U$  is even causing the second term also to be even, thereby proving the theorem. We now determine  $U$ . It is equal to the total number of nodes if the  $m^{\text{th}}$  level was full minus the actual number of nodes.

$$U = \sum_{j=1}^m D(D-1)^{j-1} - N + 1$$

$$= \frac{D[(D-1)^m - 1]}{D-2} - N + 1$$

$$= \frac{D(D-1)^m - D + D - 2}{D-2} - N$$

$$= \frac{D(D-1)^m - 2}{D-2} - N.$$

Since  $D$  is odd,  $N$  must be even (Corollary 4.1.2).  $D(D-1)^m$  is even for all  $m \geq 1$ . Since  $D-2$  is odd and  $U$  is an integer, the first term is even. Therefore  $U$  is even.

Q.E.D.



From the definitions just introduced we see that Moore graphs are generalized Moore graphs, which in turn are pseudo generalized Moore graphs which themselves are regular graphs. We now determine lower bounds for the diameter and average path length in regular graphs.

#### 4.3.1 Some Lower Bounds in Regular Graphs

Cerf, et al, [Cerf74b, Cerf74c] have derived expressions for the lower bound on the diameter and average path length in regular graphs. They do so by analyzing the tree of a pseudo generalized Moore graph of degree  $D$  with  $N$  nodes. Note that for some  $D$  and  $N$ , regular graphs may not exist, but a minimum height tree does and so the analysis is valid.

The lower bound on the diameter of regular graphs,  $DIA(N,D)$  is determined by finding the height of a minimum height tree connecting the  $N$  nodes of degree  $D$ .

$$DIA(N,D) = m = \left\lceil \log_{D-1} \frac{N*(D-2) + 2}{D} \right\rceil \text{ for } D > 2. \quad (4.5)$$

A plot of  $DIA(N,D)$  is shown in figure 4.5.

The sum of the shortest path lengths from the root of the tree to all the other nodes can easily be determined. We first assume that the  $m^{\text{th}}$  level is completely filled and then subtract the contribution by the nodes that are absent. The lower bound on the sum of the shortest path lengths for a regular graph,  $SPL_{\text{sum}}$ , is equal to the sum of the shortest path lengths from the root. Therefore

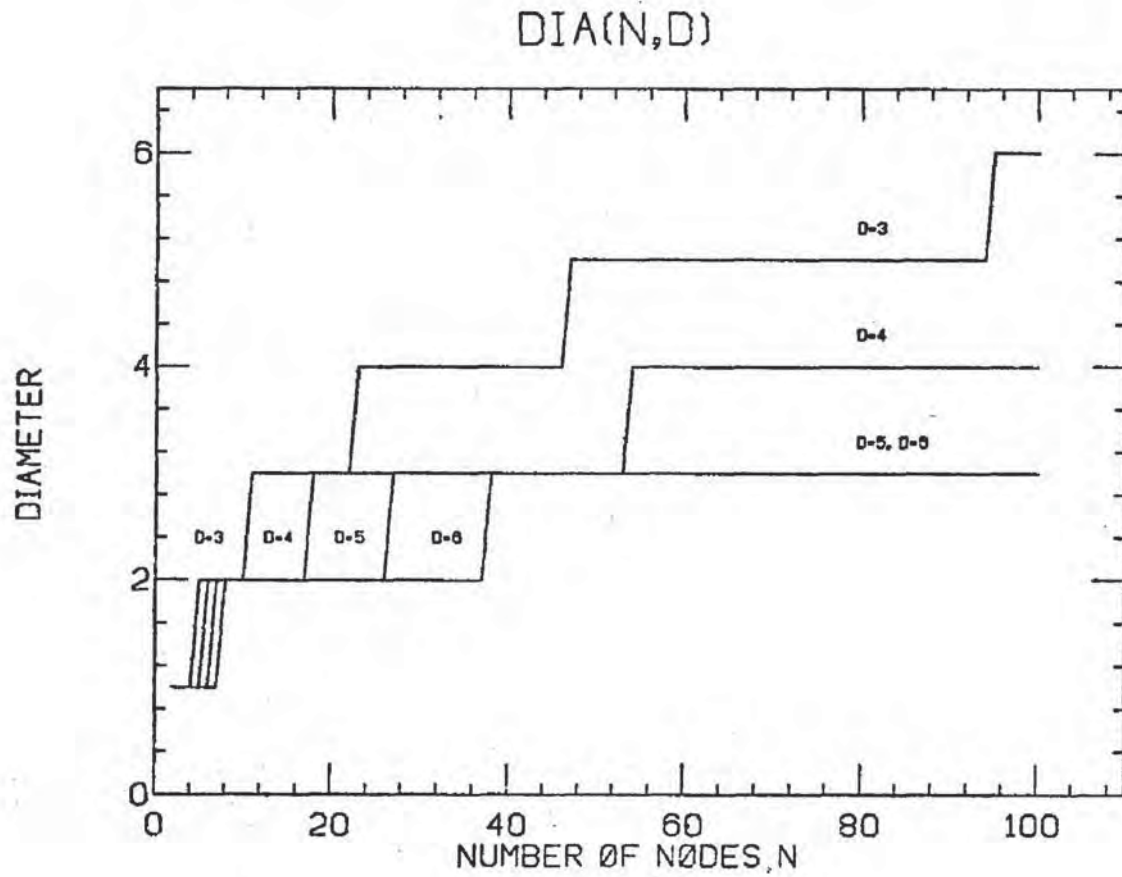


Figure 4.5. LOWER BOUND ON THE DIAMETER OF REGULAR GRAPHS.



$$\text{SPLsum}(N,D) = D * \sum_{j=1}^m j * (D-1)^{j-1} - (1 + D * \sum_{j=1}^m (D-1)^{j-1} - N) * m. \quad (4.6)$$

The dependence of SPLsum on N, and D is illustrated in figure 4.6.

The lower bound on the average shortest path length in regular graphs, SPLav, is analogously equal to the average shortest path length from the root of the tree, and is:

$$\text{SPLav}(N,D) = \frac{\text{SPLsum}(N,D)}{N-1}. \quad (4.7)$$

Figure 4.7 illustrates this relation.

These bounds are met by generalized Moore graphs, since for every node, the shortest path lengths look like the tree just described. In general these bounds will not be met by all regular graphs.

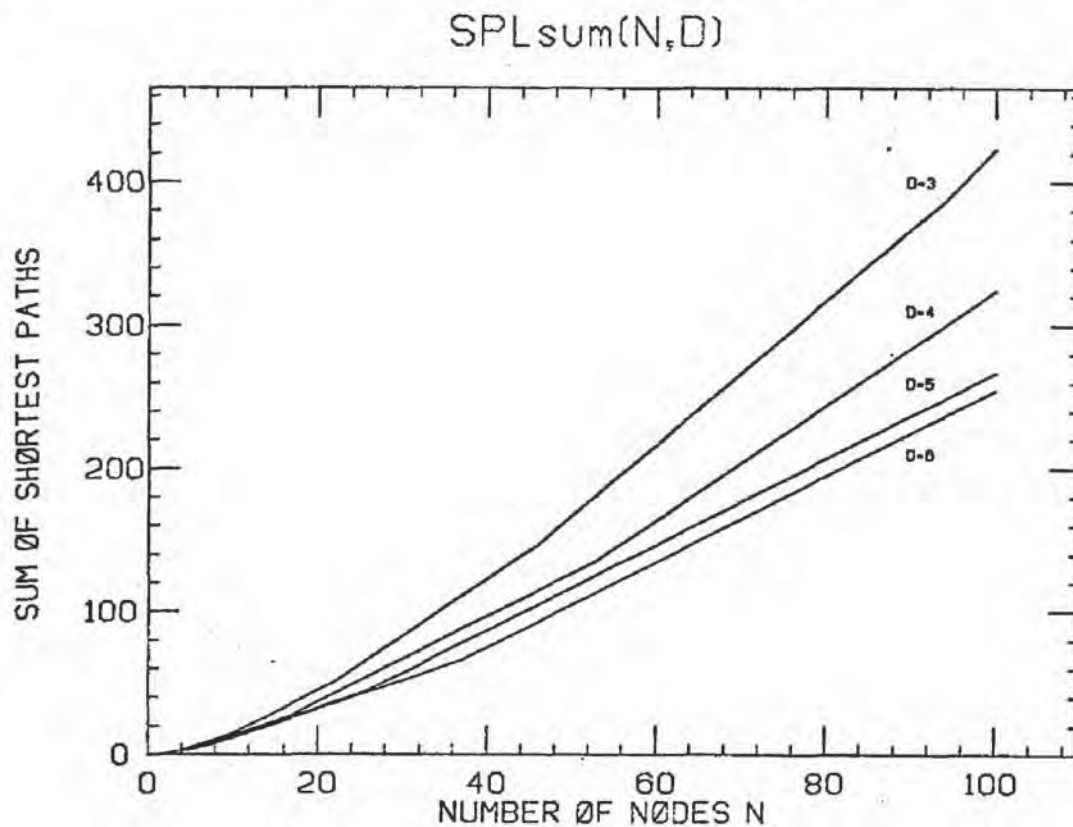


Figure 4.6. LOWER BOUND ON THE SUM OF SHORTEST PATH LENGTHS IN REGULAR GRAPHS.

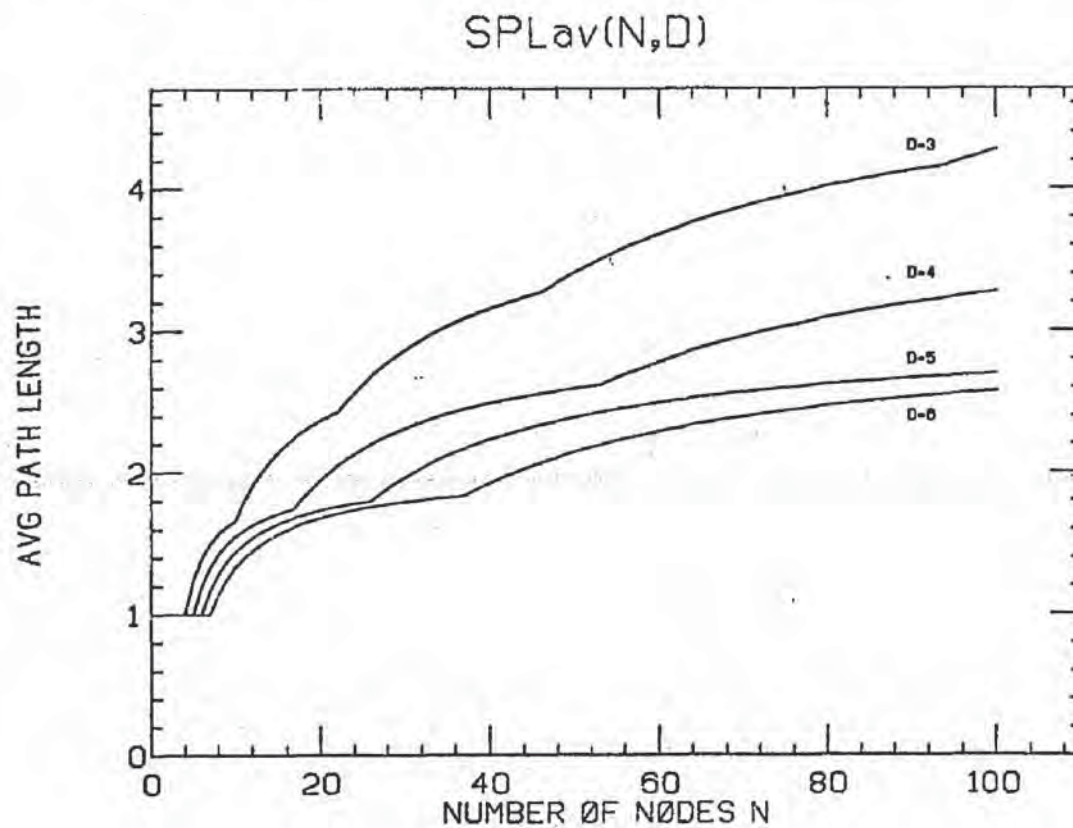


Figure 4.7. LOWER BOUND ON THE AVERAGE PATH LENGTH IN REGULAR GRAPHS.

#### 4.4 Separately Addressed Packets (SAP)

In this section we determine lower bounds\* for NPT, BDav, BDmax and BC for regular graphs when the broadcast routing strategy is to transmit separately addressed packets from the source to each destination. We assume that the routing of the packets is along the shortest path (in the graph theoretical sense) from the broadcaster to the destinations.

##### 4.4.1 Number of Packets Transmitted

Assume that each broadcast message consists of a single packet. Hence the number of packet-hops from a broadcaster to a receiver is equal to the path length from the broadcaster to the receiver. Therefore a lower bound on the number of packets transmitted to achieve the broadcast is equal to the lower bound on the sum of the path lengths. Therefore

$$NPT(N,D)_{SAP} = SPLsum(N,D). \quad (4.8)$$

The dependence of  $NPT_{SAP}$  on N and D is illustrated in figure 4.6.

##### 4.4.2 Broadcast Delay

In order to determine the lower bound on the various broadcast delays when transmitting separately addressed packets, we must determine the order in which messages (packets) are transmitted from the

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\*Since lower bounds are being determined, 1, will be dropped from  $NPT(1)$ ,  $BDav(1)$  and  $BDmax(1)$ .



broadcaster to the various receivers so that the maximum broadcast delay is minimized. Since the lower bounds are being determined by considering the root node of a pseudo generalized Moore graph, the optimal paths lie along the branches of the tree connecting the root node to the various other nodes. Figure 4.8a illustrates such a minimum height tree for a node X. The subtrees of X form the primary subtrees of this tree. We assume that X can be transmitting packets to all primary subtrees at once, and hence we must consider the delays for a particular primary subtree - the longest one. Also assume that no other traffic is present to interfere with the broadcast packets.

The rate at which packets may be transmitted to nodes in a primary subtree is determined by the transmission delays of the communication link. Let us assume that this is a constant and that all times are relative to this quantity. In order to minimize the time taken to broadcast the packets to all nodes of the primary subtree, the root X must transmit packets to the node farthest away first and so on. This ensures that there will be as many packets as possible in transit at the same time, thus minimizing the total broadcast delay. If node X started transmitting a packet at time zero, and then one more at every time interval, then the last packet would be transmitted at time equal to the number of nodes in the primary subtree minus one. This packet would arrive at its destination in one time unit, since its destination is the closest node - the root of the primary subtree. Hence the time taken for all nodes in the primary subtree to get the broadcast message is equal to the number of nodes in the primary subtree. Figure 4.8b

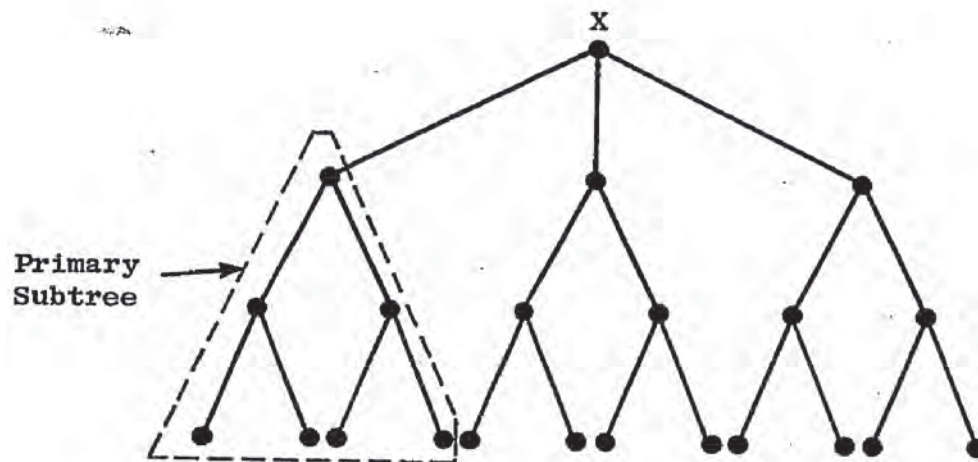


Figure 4.8a. SHORTEST PATH TREE FOR ROUTING.

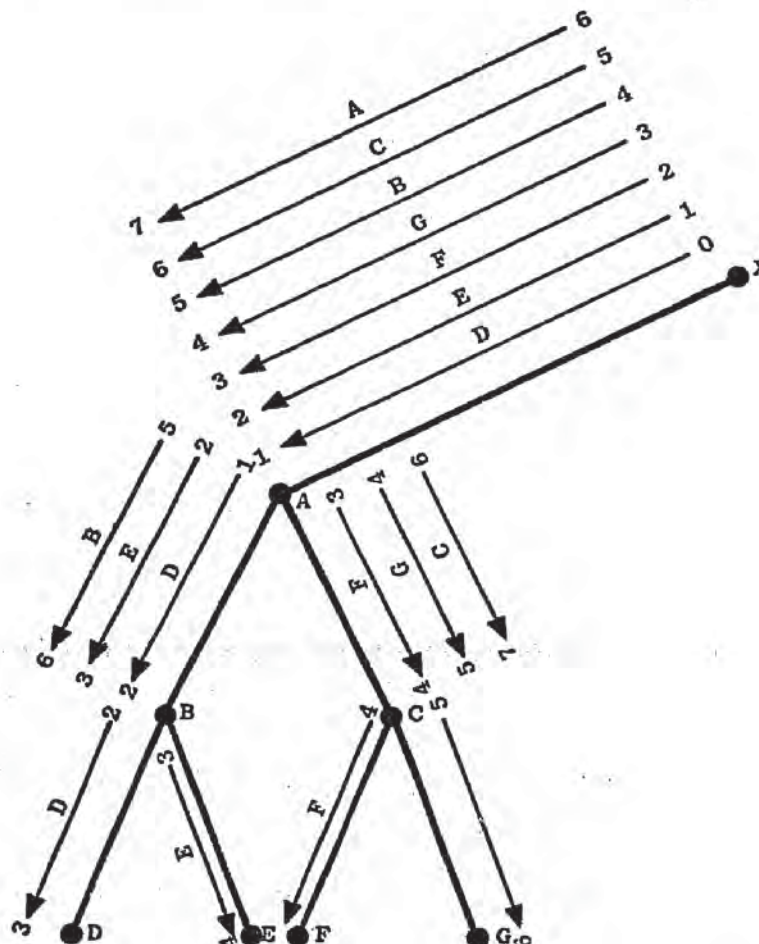


Figure 4.8b. ORDER IN WHICH PACKETS ARE TRANSMITTE



illustrates the order in which the packets are transmitted for the largest primary subtree of the tree in figure 4.8a.

We now proceed to find the number of nodes in the largest primary subtree of a pseudo generalized Moore graph.

Let  $R$  be the number of nodes remaining in the unfilled level, should there be one, and let  $h$  be the highest numbered completely filled level.

$$\begin{aligned} h &= m-1 && \text{if } U \neq 0 \\ &= m && \text{if } U = 0 \end{aligned}$$

and 
$$R = N-1 - \sum_{j=1}^h D(D-1)^{j-1}.$$

If  $R$  is equal to zero, then the number of nodes in each primary subtree is the same and equal to

$$\sum_{j=1}^m (D-1)^{j-1}.$$

However,  $R$  may not be equal to zero, and since we want to find the lower bound on delay, we want to determine the minimum number of nodes in the largest primary subtree. Hence the  $R$  remaining nodes (and hence, the  $U$  unfilled places) must be distributed as equally as possible among the primary subtrees. The minimum number of unfilled places any primary subtree can have is  $\lfloor U/D \rfloor$ . Therefore the minimum number of nodes in the largest primary subtree, and hence  $BD_{\max_{\text{SAP}}}$  is



$$BD_{\max}(N,D)_{SAP} = \sum_{j=1}^m (D-1)^{j-1} - \lfloor U/D \rfloor \quad (4.9)$$

and

$$BC(N,D)_{SAP} = N \cdot BD_{\max}(N,D)_{SAP}. \quad (4.10)$$

These relationships are illustrated in figures 4.9, and 4.10 respectively.

In order to find the average broadcast delay we must find the delay to all nodes and then divide by the number of receivers. For the moment assume that all primary subtrees are completely filled, (later we will remove this assumption). It was shown that packets were transmitted from the root at times  $0, 1, 2, \dots, (\sum_{j=1}^m (D-1)^{j-1} - 1)$  if the highest level is completely filled. These packets are first destined for nodes in level  $m$ , then  $m-1, \dots$  and finally level 1. Hence the time taken for a packet to reach its destination is given by the time at which it was transmitted plus the level number in which the destination lies. Let NPS be the number of nodes in a completely filled primary subtree.

$$NPS = \sum_{j=1}^m (D-1)^{j-1}.$$

The sum of delays for all  $D$  completely filled primary subtrees is:

$$D \cdot \sum_{j=1}^m \sum_{i=LB}^{UB} (j + NPS - i).$$

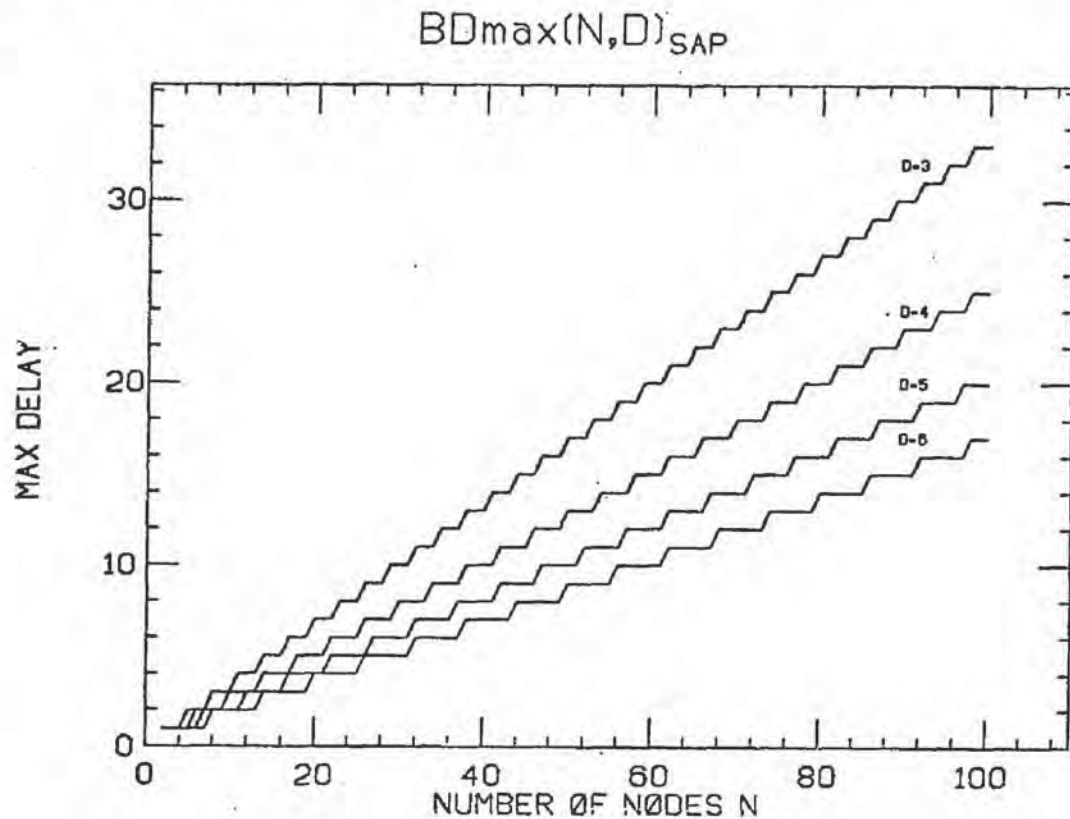


Figure 4.9. LOWER BOUND ON MAXIMUM BROADCAST DELAY FOR SEPARATELY ADDRESSED PACKETS.

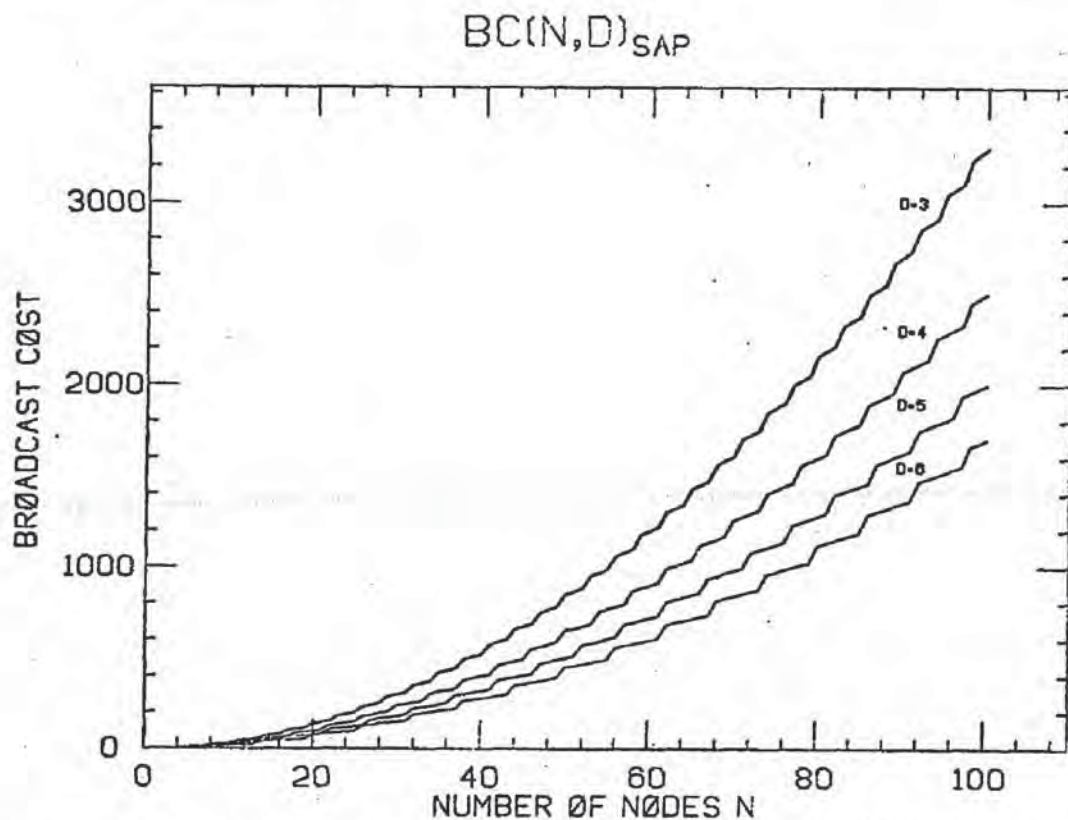


Figure 4.10. LOWER BOUND ON BROADCAST COST FOR SEPARATELY ADDRESSED PACKETS.



where

$$LB = 1 + \sum_{k=1}^{j-1} (D-1)^{k-1}$$

and

$$UB = \sum_{k=1}^j (D-1)^{k-1}.$$

The outer summation cycles through each of the levels, while the inner summation cycles through the nodes within a level. The limits of the inner summation are themselves summations dependent on  $j$  since the number of nodes in a level depends on the level number. The simplification of the summation can be found in Appendix D, and we just state the result here:

$$D*NPS*\frac{(NPS-1+2m)}{2} - D*\frac{(NPS-m)}{D-2}.$$

In general, however,  $U$  will not be equal to zero, and so the reduction in delay owing to these  $U$  unfilled places must be determined. Each unfilled place causes the reduction of one unit of delay from all nodes in its primary subtree except those at its own level. This is readily apparent since the root is able to transmit all the packets in a shorter time. This reduction amounts to:

$$U*\sum_{j=1}^{m-1} (D-1)^{j-1}.$$

The reduction in total delay owing to the  $U$  unfilled places themselves must now be determined. These unfilled positions were equally



distributed among the primary subtrees. Therefore we must cycle through the primary subtrees removing the node with the largest delay each time.

The largest delay a node in a primary subtree can have is:

$$m + (D-1)^{m-1} - 1.$$

Therefore the reduction in total delay is

$$\begin{aligned} & \sum_{i=0}^{U-1} (m + (D-1)^{m-1} - 1 - \lfloor i/D \rfloor) \\ &= U * \{ (m-1) + (D-1)^{m-1} \} - \sum_{i=0}^{U-1} \lfloor i/D \rfloor. \end{aligned}$$

The number of complete cycles through the  $D$  primary subtrees is  $\lfloor U/D \rfloor$ . During each one of these cycles,  $D$  nodes are removed; finally leaving  $U - D * \lfloor U/D \rfloor$  nodes.

$$\begin{aligned} \therefore \sum_{i=0}^{U-1} \lfloor i/D \rfloor &= \sum_{j=1}^{\lfloor U/D \rfloor} D(j-1) + \lfloor U/D \rfloor * \{ U - D * \lfloor U/D \rfloor \} \\ &= (D/2) * \lfloor U/D \rfloor * \{ \lfloor U/D \rfloor - 1 \} + U * \lfloor U/D \rfloor - D * \lfloor U/D \rfloor^2 \\ &= \lfloor U/D \rfloor * \{ (D/2) * \lfloor U/D \rfloor - (D/2) + U - D * \lfloor U/D \rfloor \} \\ &= \lfloor U/D \rfloor * \{ U - (D/2) * \lfloor U/D \rfloor - (D/2) \}. \end{aligned}$$

$$\therefore \text{Total Delay} = D \cdot \text{NPS} \cdot (\text{NPS} - 1 + 2m) / 2 - D \cdot (\text{NPS} - m) / (D - 2)$$

$$- U \cdot \sum_{j=1}^{m-1} (D-1)^{j-1} - U \cdot \{ (m-1) + (D-1)^{m-1} \}$$

$$- \lfloor U/D \rfloor \cdot \{ U - (D/2) \cdot \lfloor U/D \rfloor - (D/2) \}.$$

$$\therefore \text{BDav}(N, D)_{\text{SAP}} = \text{Total Delay} / (N - 1). \quad (4.11)$$

This relation is shown in figure 4.11.

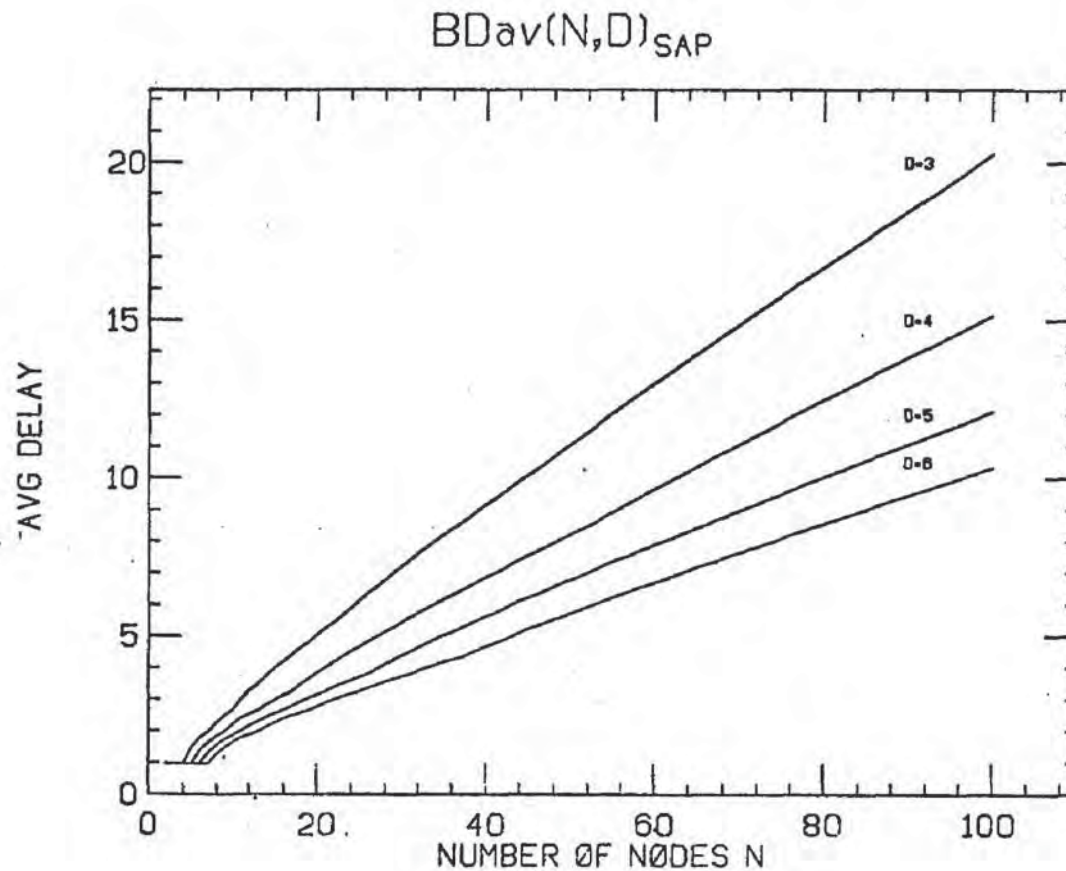


Figure 4.11. LOWER BOUND ON AVERAGE BROADCAST DELAY FOR SEPARATELY ADDRESSED PACKETS.



#### 4.5 Multi-Destination Addressing (MDA) and Source Based Forwarding (SBF)

This section determines the performance measures when the broadcast routing technique is either multi-destination addressing or source based forwarding. The two techniques have the same performance since they both forward the packets along the optimal paths from the broadcaster to the receivers.

The number of packets transmitted is equal to the minimum number of edges by which the broadcaster is connected to the all the destinations. This is  $N-1$ . Therefore

$$NPT(N,D)_{MDA} = NPT(N,D)_{SBF} = N-1. \quad (4.12)$$

From the view point of the node initiating the broadcast the rest of the network looks like a minimum height tree. This tree is similar to the tree of a pseudo generalized Moore graph. Therefore we have:

$$BDav(N,D)_{MDA} = BDav(N,D)_{SBF} = SPLav(N,D) \quad (4.13)$$

$$BDmax(N,D)_{MDA} = BDmax(N,D)_{SBF} = DIA(N,D) \quad (4.14)$$

$$BC(N,D)_{MDA} = BC(N,D)_{SBF} = N * BDmax(N,D)_{MDA}. \quad (4.15)$$

These relationships are illustrated in figures 4.7, 4.5, and 4.12 respectively.

We have not determined the number of packet copies transmitted and the various delays if a restricted multi-destination addressing scheme

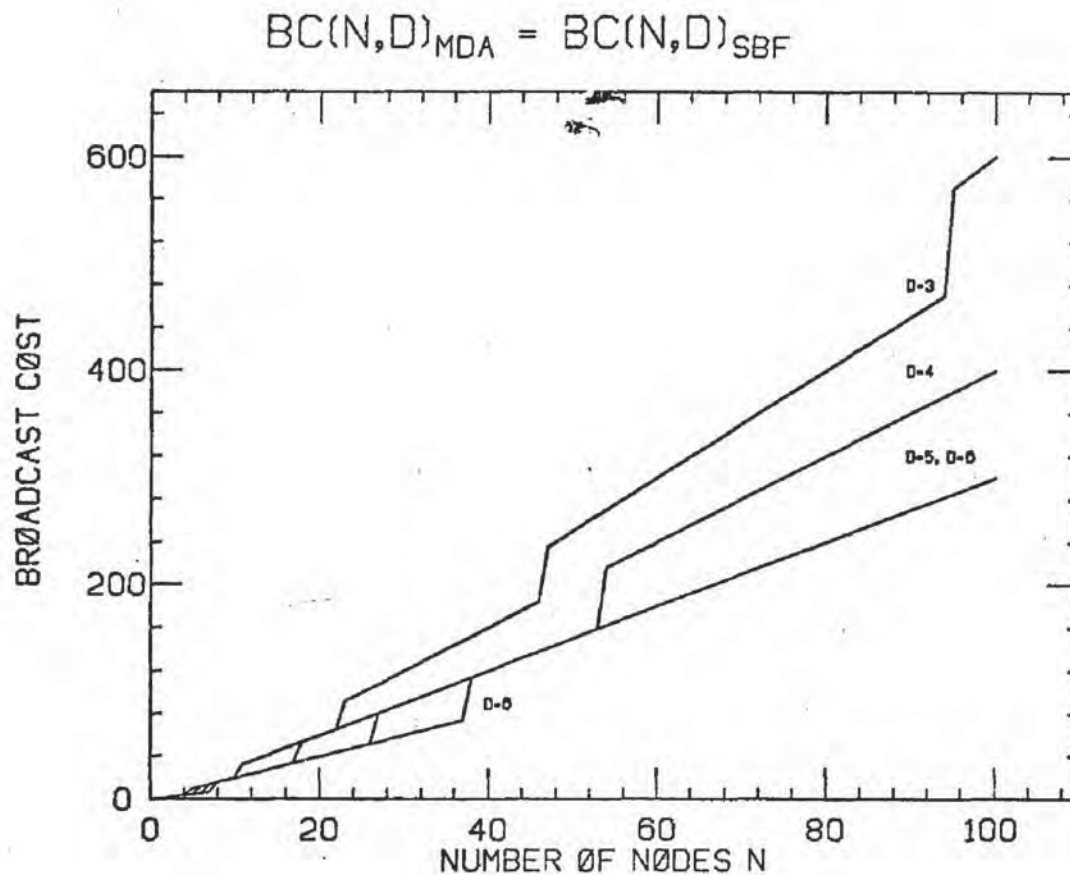


Figure 4.12. LOWER BOUND ON BROADCAST COST FOR MULTI-DESTINATION ADDRESSING AND SOURCE BASED FORWARDING.

were to be used. This is a subject for future research. The analysis parallels that used to determine the performance measures if separately addressed packets were transmitted (cf. section 4.4). One must determine the optimal order in which destinations are assigned, to the restricted multi-destination address field in the header, and the optimal order in which the packets are transmitted. This formulation would then include the analysis for separately addressed packets and multi-destination addressing as special cases, where the length of address field is one, or as long as desired, respectively.