

“Evolution” by vertex of even-order regular graphs

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In graph theory circles, it is generally interesting to examine a structurally similar problem, but from a different point of view, to see alternate forms of graphs under certain transformations.

For example,  $\Gamma_n$ , the  $n$ -vertex complete graph, can be constructed from  $\Gamma_{n-1}$ , the complete graph on  $n-1$  vertices, by adding a vertex and connecting it to each vertex of  $\Gamma_{n-1}$ . The same transformation on any other regular graph of other degree does not achieve the goal. Pal Erdős and Alfréd Rényi addressed graph evolution in [3] from another perspective.

In the present work we are only concerned with undirected, even-order regular graphs.

In the first section of this paper we define the evolutionary transformation and prove several related theorems. The second and third sections concern graphs for which the transformation cannot be applied.

Definition 1. Let  $\Gamma = (P, E)$  be an  $n$ -vertex simple graph and  $p_i \in P$ ,  $p_j \in P$  be neighboring vertices, and  $p$  be a vertex which is a neighbor to neither  $p_i$  nor  $p_j$ . ( $P$  and  $E$  represent the sets of vertices and edges of the graph  $\Gamma$ .)

We take the EC transformation on vertex  $p$  to be the following:

$$(1) \quad EC: E \rightarrow E'$$

$$\text{where} \quad E' = E \setminus \{(p_i p_j)\} \cup \{(pp_i), (pp_j)\}$$

Herein we designate the set of all even regular graphs using as  $\mathbf{PR}$  and the sets of all 2, 4, ...,  $k$  ( $k$  even) order regular graphs as  $\mathbf{PR}_2, \mathbf{PR}_4, \dots, \mathbf{PR}_k$ . Then the following must be the case:

1.  $\mathbf{PR} = \mathbf{PR}_2 \cup \mathbf{PR}_4 \cup \dots \cup \mathbf{PR}_k \cup \dots$
2.  $\forall i \neq j, (i, j \text{ even}), \mathbf{PR}_i \cap \mathbf{PR}_j = \emptyset$
3.  $\forall \Gamma_{n,k} \in \mathbf{PR}_k, \exists ! \mathbf{PR}_k: \Gamma_{n,k} \in \mathbf{PR}_k$ ; that is, the set  $\mathbf{PR}$  has a classification.

Definition 2. Let  $\Gamma_{n,k} \in \mathbf{PR}_k$  be an  $n$ -vertex,  $k$ -regular graph. Further suppose  $\Gamma_{n,k} = (P, E)$  and  $\Gamma_{n+1} = (P \cup \{p\}, E')$ , where  $p \notin P$ .

We take the ET transformation on vertex  $p$  to be the  $k/2$  EC transformation on vertex  $p$ , that is, let  $(p_1 p_2), (p_3 p_4), \dots, (p_{k-1} p_k)$  be edges in  $\Gamma_{n,k}$ , then

$$(2) \quad ET: \Gamma_{n,k} \rightarrow \Gamma_n$$

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$$\text{where } E' = E \setminus \{(p_1 p_2), (p_3 p_4), \dots, (p_{k-1} p_k)\} \cup \{(pp_1), (pp_2), \dots, (pp_k)\}$$

Theorem 1. Let  $k$  be an even number. The ET transformation can be used for every  $\Gamma_{n,k} \in \mathbf{PR}_k$ .

PROOF. It is sufficient if we observe that in every  $\Gamma_{n,k}$  there exist  $k/2$  independent edges. For the proof we use the following theorem from G. A. Dirac [2]:

“If in a simple graph every vertex has degree of at least  $r$  ( $r > 1$ ), then there exists a cycle in the graph of length at least  $r + 1$ .”



If we take  $K$  to be a cycle of length  $k+1$  in  $\Gamma_{n,k}$  (one certainly exists on the basis of the theorem cited), then using the method illustrated in figure 1 to select the edges  $(p_1p_2), (p_3p_4), \dots, (p_{k-1}p_k)$  we get a set of exactly  $k/2$  unconnected edges.

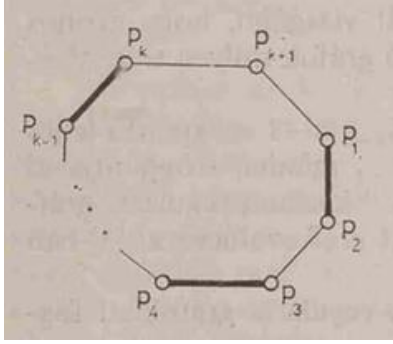


Figure 1

Definition 3. Let  $\Gamma_{n,k} = (P, E) \in \mathbf{PR}_k$ ,  $p \in P$ , and we take the vertex pairs  $p_1p_2, p_3p_4, \dots, p_{k-1}p_k$  to be neighbors of  $p$  but pairwise not neighbors. Then we take the ET transformation to be:

$$(3) \text{ ET}^{-1}: \Gamma_{n,k} \rightarrow \Gamma_{n-1} = (P', E'), \quad P' = P \setminus \{p\} \\ E' = E \setminus \{ (pp_1), (pp_2), \dots, (pp_k) \} \cup \{ (p_1p_2), (p_3p_4), \dots, (p_{k-1}p_k) \}$$

Definition 4. Let  $\Gamma_n = (P, E)$  be a graph on  $n$  vertices,  $p \in P$ , and  $q_1, q_2, \dots, q_r$  be neighbors of  $p$  in  $\Gamma_n$ . Then we take the subgraph of  $\Gamma_n$  generated by  $p$  (designate this  $G_p = (P', E')$ ), and it turns out this is the subgraph where  $P' = \{q_1, q_2, \dots, q_r\}$  and  $(q_i, q_j) \in E' \Leftrightarrow (q_i, q_j) \in E$ .

Definition 5. Let  $\Gamma_n \in \mathbf{PR}$  and  $p \in \Gamma_n$ . Then the vertex  $p$  is T-characterized if the complement of the subgraph on  $\Gamma_n$  generated by it contains a 1-factor.

Theorem 2. The  $\text{ET}^{-1}$  transformation can be applied to a graph  $\Gamma_{n,k} = (P, K) \in \mathbf{PR}_k$  if and only if there exists a T-characterized vertex  $p \in \Gamma_{n,k}$ .

PROOF. Necessity: From the definition of  $\text{ET}^{-1}$  it follows that there exists  $p \in \Gamma_{n,k}$  from which we can select neighbors  $p_1p_2, p_3p_4, \dots, p_{k-1}p_k$  which are pairwise not neighbors. In the complement of the subgraph on  $\Gamma_n$  constructed on  $p_1, p_2, \dots, p_k$  they will be neighbors, and they comprise a 1-factor, as any two vertex pairs have no common vertex. This is precisely what it means for the vertex  $p$  to be T-characterized.

Sufficiency: The proof in this direction is the exactly the reverse of the above necessity proof.

Theorem 3. For every even  $k \geq 4$  there is a graph  $\Gamma_{n,k} \in \mathbf{PR}_k$  which has no vertex  $p$  which is T-characterized.

Equivalent formulation: For every even  $k \geq 4$  there exists a graph  $\Gamma_{n,k} \in \mathbf{PR}_k$  for which no graph  $\Gamma_{n-1,k} \in \mathbf{PR}_k$  can be constructed with the ET transformation.

PROOF. Let  $k$  be an arbitrary even number,  $k \geq 4$ . Let us examine two complete graphs on  $k+1$  vertices  $\Gamma_{k+1,k}^1 = (P_1, E_2)$  and  $\Gamma_{k+1,k}^2 = (P_2, E_2)$ . Let  $(p_{i1}, p_{j1}) \in E_1$  and  $(p_{i2}, p_{j2}) \in E$ . Then we construct the following

graph  $\Gamma_{n,k} = (P, E)$

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$$(4) \quad P = P_1 \cup P_2$$

$$(5) \quad E = E_1 \cup E_2 \setminus \{ (p_{i1}, p_{j1}), (p_{i2}, p_{j2}) \} \cup \{ (p_{i1}, p_{i2}), (p_{j1}, p_{j2}) \}$$

The graph  $\Gamma_{n,k}$  where for  $k = 4$  is illustrated in figure 2.

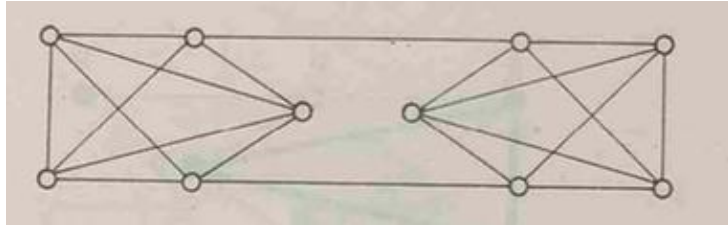


Figure 2

We will demonstrate that  $\Gamma_{n,k}$  constructed in this way has does not have a T-characterized vertex.

As every graph  $\Gamma_{n,k}$  constructed in this way is symmetric, for the proof it is enough to demonstrate for each vertex of the subgraph (for example  $\Gamma_{k+1,k}^1$ ). Here we can differentiate between two cases of selected pairs of vertices:

- a)  $p_{i1}, p_{j1}$
- b) any other pair

For the case in a), we take  $G_1 = (A_1, B_1)$  to be the subgraph constructed on neighbors of  $p_{i1}$ . Then:

$$(6) \quad A_1 = P_1 \cup \{p_{i2}\} \setminus \{p_{i1}\}$$

$$(7) \quad B_1 = E_1 \setminus \{ (p_{i2}p_r) \mid \forall p_r \in P_1 \setminus \{p_{i1}\} \}$$

So the complement  $\bar{G}_1 = (A'_1, B'_1)$  is therefore the following:

$$(8) \quad A'_1 = A_1$$

$$(9) \quad B'_1 = E_1 \setminus \{ (p_{i2}p_s) \mid \forall p_s \in P_1 \setminus \{p_{i1}\} \}$$

So  $\bar{G}_1$  does not contain a 1-factor, that is,  $p$  is not T-characterized (and similarly  $p_{i2}, p_{j1}, \dots, p_{j2}$ ). For the case in b), let  $p \in P_1, p \neq p_{i1} \neq p_{j1}$ , and  $G_2 = (A_2, B_2)$  be the subgraph constructed on the neighbors of  $p$ :

$$(10) \quad A_2 = P_1 \setminus \{p\}$$

$$(11) \quad B_2 = E_2 \setminus \{ (p_{i1}p_{j1}), (pp_r) \mid \forall p_r \in P_1 \setminus \{p\} \}$$

Then  $\bar{G}_2 = (A', B')$  will be the following:

$$(12) \quad A'_2 = A_2$$

$$(13) \quad B'_2 = \{ (p_{i1}p_{j1}) \}$$

So  $\bar{G}_2$  does not contain a 1-factor, that is, no vertex in case b) is T-characterized. And so the 2 theorem is proved.

In figure 3/a-d the graphs  $G_1, \bar{G}_1, G_2, \bar{G}_2$  are illustrated for  $k = 4$ . (The bolded edges and vertices belong to the corresponding graphs.)

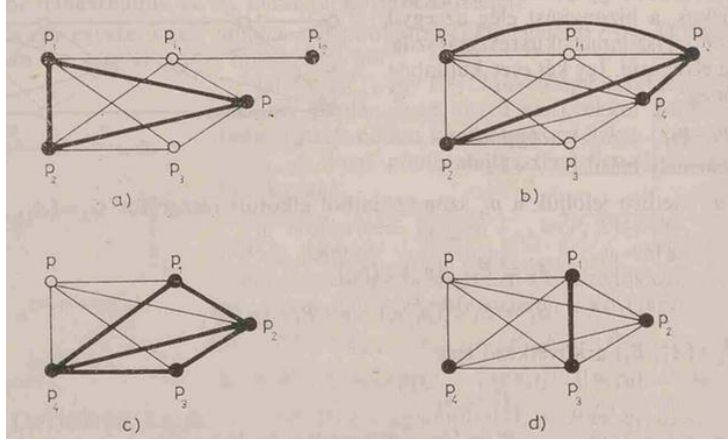


Figure 3

6. Definition. We define a Vertex Prime (VP) graph as a graph in  $\mathbf{PR}$  which contains T-characterized vertex.

Then Theorem 3 can be formulated as follows: Each of the classes of graphs  $\mathbf{PR}_4, \mathbf{PR}_6, \dots, \mathbf{PR}_k, \dots$  contains a VP graph.

In the following we examine the cases under which a vertex is not T-characterized. We take advantage of the following well-known result [1].

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Lemma 1. A complete graph  $\Gamma_n$  (where  $n$  is even) can be expanded into  $n-1$  one-degree factors.

Theorem 4. Let  $\Gamma_{n,k} \in \mathbf{PR}_k$  and  $p \in \Gamma_{n,k} = (P, E)$ . If  $p$  is not T-characterized, then there exist at least  $k-1$  cycles of length 3 which contain  $p$ .

PROOF. If  $p$  is not T-characterized, then the complement of the subgraph of  $\Gamma_{n,k}$  generated by it ( $G = (A, B)$ ) does not contain 1-factor. By Lemma 1,  $\bar{G}$  contains  $n-1$  edge-independent 1-factors, and in  $G$  from these factors an edge must exist such that in  $\bar{G}$  they are not a 1-factor. So the number of edges in  $G$  is at least  $k-1$ . As it is the case in  $G$  that every edge endpoint is a neighbor of  $p$ , a 3-cycle containing  $p$  is thus constructed. And hence the theorem is proved.

For  $k = 4$ , the following illustrations in figure 4/a-h depict cases where the vertex  $p$  is not T-characterized (bold edges indicate  $\bar{G}$ ).

Lemma 2. Let  $\Gamma_n$  be an  $n$ -vertex complete graph ( $n$  an even number) and we denote the number of all 1-factors of  $\Gamma_n$  as  $F$ . Then

$$(14) \quad F_n = \frac{\binom{n}{2} \binom{n-2}{2} \dots \binom{2}{2}}{\left(\frac{n}{2}\right)!}$$

PROOF. It is easy to see that the assertion is true for the case  $n = 2$ . Suppose it also holds for any

$k \leq n - 2$ , that is:

$$(15) \quad F_k = \frac{\binom{k}{2} \binom{k-2}{2} \dots \binom{2}{2}}{\binom{k}{2}!}$$

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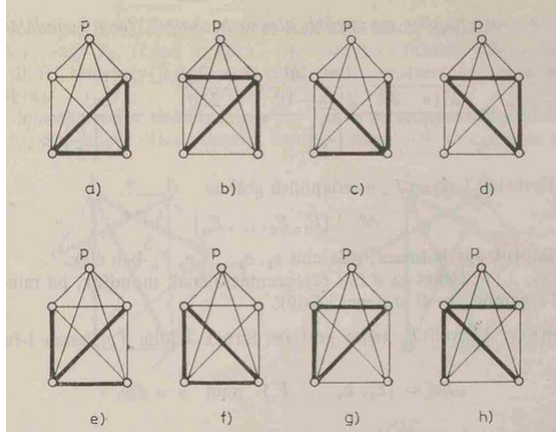


Figure 4

Then  $\Gamma_k$  has two vertices beyond which  $k+1$  new vertex pairs can be selected (we don't take note of the ordering of the two points), each of which has an  $F_k$  numbered 1-factor which each share exactly one edge, so

$$(16) \quad F_{k+2} = (k+1) \cdot F_k$$

Recalling that

$$(17) \quad \frac{\binom{k+2}{2}}{\frac{k+2}{2}} = \frac{2 \cdot (k+2)!}{2! \cdot k! \cdot (k+2)} = k+1$$

So thus

$$(18) \quad F_{k+2} = \frac{\binom{k+2}{2}}{\frac{k+2}{2}} * \frac{\binom{k}{2} \binom{k-2}{2} \dots \binom{2}{2}}{\frac{k}{2}!} = \frac{\binom{k+2}{2} \binom{k}{2} \dots \binom{2}{2}}{\frac{k+2}{2}!}$$

This concludes the proof of the theorem.

Corollary.

4. In the context of (16) we can write  $F_n$  recursively as follows:

$$(19) \quad F_n = (n-1) \cdot F_{n-2}$$

2. The proof of Lemma 2 shows that if  $e$  is an edge contained in an  $E$  1-factor of  $\Gamma_n$ , then exactly  $F_{n-2}$  such

1-factors are in  $\Gamma_n$  which with  $E$  contain exactly the  $e_i$  shared edges.

3.If the number of distinct 1-factors in  $\Gamma_n$  is denoted  $FD_n$ , then based on Lemma 1 and (19) we have:

$$FD_n = F_n \Leftrightarrow n-1 = (n-1) \cdot F_{n-2} \Rightarrow 1=F_{n-2} \Rightarrow n=4$$

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4. The correspondence to (14) can be simplified by hiding the numerator as follows:

$$F_n = \frac{n!}{\frac{2!(n-2)!}{\frac{n!}{2}} \cdot \frac{2!(n-4)!}{\frac{n!}{2}} \cdots \frac{2!0!}{\frac{n!}{2}}} = \frac{n!}{2^{n/2} \frac{n!}{2}}$$

Definition 7. Let  $\Gamma_n$  be a graph with  $n$  vertices and let

$$(20) \quad E = \{ E_1, E_2, \dots, E_m \}$$

be a set of 1-factors of  $\Gamma_n$  and  $e_1, e_2, \dots, e_p$  be edges of  $\Gamma_n$ .

We say the edges  $e_1, e_2, \dots, e_p$  are a representation of  $E$  if every 1-factor of  $E$  shares at least one edge with  $\{ e_1, e_2, \dots, e_p \}$ .

Lemma 3. Let  $\Gamma_n$  be a complete graph ( $n$  even). Denote the set of all 1-factors of  $\Gamma_n$  as

$$(21) \quad E = \{ E_1, E_2, \dots, E_r \} \text{ where } r = F_n \quad \text{and}$$

$$(22) \quad R = \{ e_1, e_2, \dots, e_{n-1} \}$$

a set of edges of  $\Gamma_n$ . Then  $R$  is a representation of  $E$  if and only if for any two  $e_i \neq e_j$ , there does not exist  $E_k \in E$  for which  $e_i \in E$  and  $e_j \in E$ .

PROOF. Necessity: Suppose that  $R$  is a representation of  $E$  and there exist  $e_i \neq e_j$  and  $E_k$  where  $e_i \in E_k$  and  $e_j \in E_k$ . On the basis of Lemma 2, Corollary 2,  $e_i$  represents an  $F_{n-2}$  1-factor of  $E$ , and  $e_j$  does also. As both edges are in  $E_k$ , it is the case that the for the 1-factor  $K_{ij}$  represented by the two edges, it holds that

$$(23) \quad K_{ij} < 2 \cdot F_{n-2}$$

If we suppose that the remaining  $n-3$  edges in  $R$  represent the maximum  $n-1$  1-factors, then it is the case that for the  $K_E$ , the elements of  $E$  represented by  $R$ , it holds that

$$(24) \quad K_E < (n-1) \cdot F_{n-2}$$

which is a contradiction, as the number of elements in  $E$  is  $(n-1) \cdot F_{n-2}$  and  $R$  is a representation of  $E$ , so for  $K_E$  the equality must hold.

Sufficiency: As every  $E_k \in E$  1-factor contains only one edge  $e_i \in R$ , and on the basis of Lemma 2, Corollary 2, an edge  $e_i$  represents an  $F_{n-2}$  1-factor of  $E$ , since there are  $n-1$  of  $R$  numbering  $(n-1) \cdot F_{n-2}$ , this is all of the edges of  $E$ .

We have proved the theorem.

Theorem 5. For a complete graph  $\Gamma_n$  ( $n > 4$ , even), all 1-factors can be represented by exactly  $n-1$  edges if and only if those edges are joined by a vertex.

PROOF. Necessity: Let  $p \in \Gamma_n$  be such a vertex. As every 1-factor of  $\Gamma_n$  is contained by an edge attached to  $p$ , it follows that all such edges are indeed an  $n-1$  element representation of all 1-factors of  $\Gamma_n$ .

Sufficiency: Let us examine an arbitrary edge  $(p_1p_2) \in \Gamma_n$  ( $p_1$  and  $p_2$  are the endpoints of edge  $e_i$ ). Let this be the first element of the  $n-1$  element representation. By Lemma 3, the second element can only be selected from amongst such  $e_j$  for which the only 1-factor containing  $e_i$  does not also appear.

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It is easy to see that such an edge can only be selected from those which have  $p_1$  or  $p_2$  as an endpoint. (See figure 5.) (The dark thick lines are edges that may be chosen.)

Suppose that we choose  $(p_1p_3) = e_j$  as the second element of the representation (see figure 6.)

Continuing according to Lemma 3, we may choose from amongst  $p_1$  and  $p_2$  for attaching (that which can still be chosen), and immediately for the  $e_1$  edge we can also take advantage of Lemma 3,

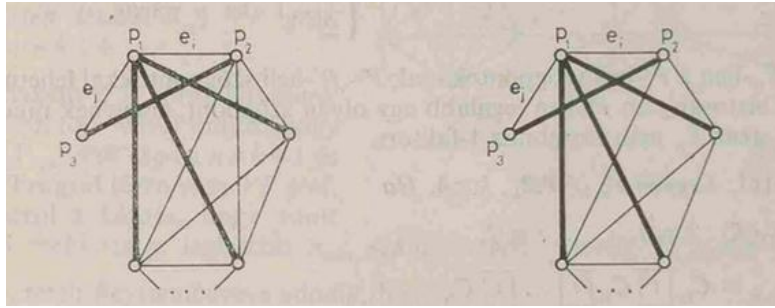


Figure 5

Figure 6

thus the desired representation for  $p_3$  must attach. Such an edge could only be  $(p_2p_3)$ . With this, the edges which can be chosen have been consumed. Thus this selection constitutes a representation if and only if  $n=4$ ; for all other cases the  $n-1$  edges which can be selected as the elements of the representation are attached to  $p_1$  (thus necessitating the condition that  $n>4$ ). We have thus proved the theorem. For  $n=4$  the entire representation is depicted in figure 4.

In the remainder of the paper we introduce a type of graph which under a certain set of assumptions proves to be a VP graph. We will examine, for a given even number  $k$ , what the smallest number  $n$  is such that  $\Gamma_{n,k} \in \mathbf{PR}_k$  is a VP graph.

Denote by  $\mathbf{C}$  the set of all complete graphs where the elements are  $C_1, C_2, \dots, C_n$ . ( $C_n$  is the complete graph of  $n$  vertices.)

Definition 8. Let  $C_{i1}, C_{i2}, \dots, C_{it}$  be (not necessarily distinct) graphs in  $\mathbf{C}$  where  $C_{ij} = (P_j, E_j)$  and  $j=1, 2, \dots, t$ . The  $t$ -composition of graphs  $C_{i1}, C_{i2}, \dots, C_{it}$  (indicated by “ $t$ ”) is a graph  $\Gamma = (P, E)$  where the following are true:

- (a)  $\Gamma$  is a regular graph
- (b)  $P = P_1 \cup P_2 \cup \dots \cup P_t$ ,  $\forall i \neq j: P_i \cap P_j = \emptyset$
- (c)  $E = E_1 \cup E_2 \cup \dots \cup E_t \cup E'$ ,  $\forall i \neq j: E_i \cap E_j = \emptyset$  where  $E'$  is the set of so-called connector edges.

COMMENTS.

1. The set  $E'$  of edges is not easily determined.
2.  $E'$  determines the connectedness of the graph  $\Gamma$ . That is, if  $E' = \emptyset$ , then  $\Gamma$  is a  $t$ -component regular graph from complete subgraphs and  $i_1 = i_2 = \dots = i_t$ .
3. If  $\Gamma = \Gamma_{n,k}$  and  $\max(i_1, i_2, \dots, i_t) = L$ , then  $L \leq k+1$ .

Theorem 6. Let  $\Gamma_n = (P, E)$  be a connected graph on  $n$  vertices. If in  $\bar{\Gamma}_n$  there exists a complete subgraph on  $\lfloor \frac{n}{2} \rfloor + 1$  vertices, then in  $\Gamma_n$  there is no 1-factor. (We will use  $\bar{\Gamma}_n$  to indicate the graph complement of  $\Gamma_n$ .)

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PROOF. Let  $G = (P', E')$  be an  $\lfloor \frac{n}{2} \rfloor + 1$  vertex complete subgraph. Then

$$|P'| = \lfloor \frac{n}{2} \rfloor + 1 \Rightarrow |P \setminus P'| = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ odd or} \\ = \lfloor \frac{n}{2} \rfloor - 1 & \text{if } n \text{ even.} \end{cases}$$

In  $\bar{\Gamma}_n$  the vertices of  $P'$  can only be connected to vertices of  $P \setminus P'$ , so certainly in  $P'$  there is at least one vertex which has no pair in  $P \setminus P'$ , that is,  $\Gamma_n$  does not contain a 1-factor.

Theorem 7. Let  $\Gamma_{n,k} \in \mathbf{PR}$  and  $k \geq 4$ . Further suppose the following:

1.  $C_{ij} \in \mathbf{C}$ ,  $j=1,2,\dots,t \geq 2$ .
2.  $\Gamma_{n,k}$  is the  $t$ -composition of  $C_{i1}, C_{i2}, \dots, C_{it}$
3.  $\frac{k+4}{2} \leq i_j \leq k$   $j=1,2,\dots,t \geq 2$

Then  $\Gamma_{n,k}$  is a VP graph.

PROOF. We shall examine an arbitrary vertex  $p$  in  $\Gamma_{n,k}$ . Then from property b) of the  $t$ -composition, it follows that there exists  $C_{ij}$  in  $\Gamma$  where  $p \in C_{ij}$  and  $i_j$  fulfills the conditions of Theorem 3. If we now examine the subgraph of  $\Gamma_{n,k}$  generated by  $p$ , there are  $k$  vertices and it contains  $C_{ij}$ , which fulfills the conditions for Theorem 6 because  $\frac{k+4}{2} \leq i \leq k$ . That is, the complement of the subgraph generated by  $p$  does not contain a 1-factor. This means that  $p$  is not  $T$ -characterized. As we did not connect an edge to  $p$ , it can be done to any vertex in  $\Gamma_{n,k}$ , so  $\Gamma_{n,k}$  is a VP graph.

A graph meeting conditions 1, 2, and 3 is shown in Figure 7.

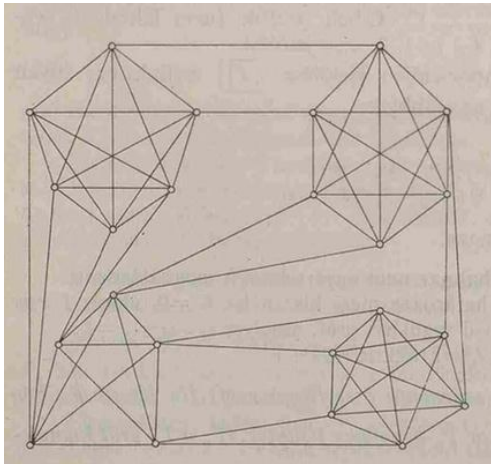


Figure 7

# COMMENTS.

1. From the proof of Theorem 7 it easily follows that to the extent that in  $\Gamma_{n,k}$  there are subgraphs  $C_{ij} \in \mathbf{C}$ , where  $i_j = \frac{k}{2} + 1$ , then  $\Gamma$  is not a VP graph. Such a graph is shown in figure 8.
2. Theorem 7 also addresses the statement of Theorem 3, thus we now know that the VP graphs constructed in Theorem 3 are not the only to exist for any even number  $k$ .
3. If we examine for the minimum case the first and third suppositions for Theorem 7, then we have  $t=2$  and  $i_1 = i_2 = \frac{k+4}{2}$ , and we get that the number of vertices is  $n = k + 4$  for the  $t$ -composition of  $C_{i1}$  and  $C_{i2} = \Gamma_{n,k}$  VP graph.

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From this follows the following theorem:

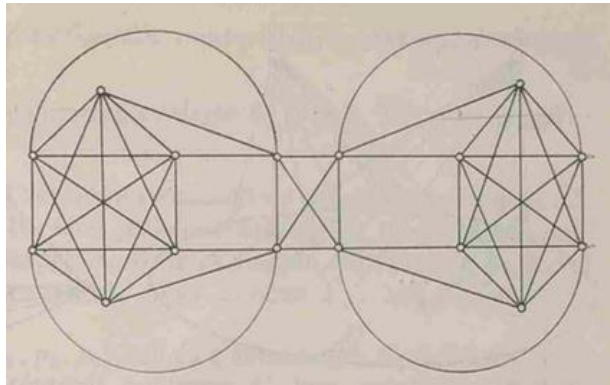


Figure 8

Theorem 8. For any even number  $k \geq 4$  there exists a  $\Gamma_{n,k}$  VP graph where  $n = k + 4$ .

As an example, in figure 9 we can see a  $\Gamma_{8,4}$  graph. As we know, for any  $\Gamma_{n,k} \in \mathbf{PR}$  where  $n \geq k + 1$  and  $\Gamma_{k+1,k}$  is complete there is no VP graph, so the question arises, for the given  $k$  cases, what is the smallest  $n_{\min}$  number for which there exists  $\Gamma_{n,k} \in \mathbf{PR}$  a VP graph.

Taking into account Theorem 8, it follows that

$$(25) \quad k+2 \leq m_{\min} \leq k + 4$$

9.Theorem. If  $\Gamma \in \mathbf{PR}$  and  $n = k+2$ , then  $\Gamma$  is not a VP graph.

PROOF. The proof can be completed using total induction on the degree. In figure 10 the case of  $k=2$  is depicted, for which explanation can be seen clearly.

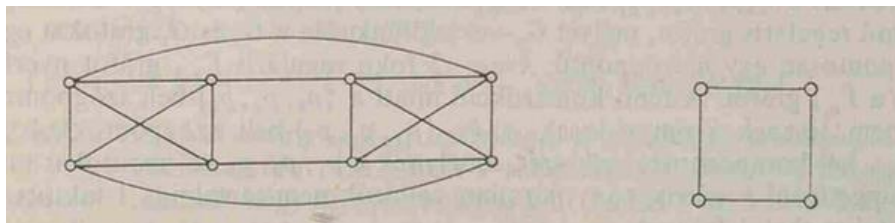


Figure 9

Figure 10

Suppose that the statement is true for  $\Gamma$ . We examine a  $\Gamma \in \mathbf{PR}$  graph where  $n = k + 2$  and select from these an arbitrary vertex  $p$ . Then it is easy to see that  $\Gamma_{n,k}$  contains exactly one other vertex  $q$  which has the same neighbors as  $p$ , that is, the subgraphs generated by  $p$  and  $q$  are the same (see figure 11). Denote as  $G_p$  the subgraph of  $\Gamma_{n,k}$  generated by  $p$  (also by  $q$ ).  $G_p$  has exactly  $k$  vertices, is a degree  $k - 2$  regular graph, which on the basis of the inductive assumption meets the conditions of the assertion, that  $G_p$  is a VP graph.

Based on this we can select any vertex of  $G_p$ , (let us call it  $z$ ), and in the complement of the subgraph of  $G$  ( $G'$ ) generated by it there is a 1-factor. Denote as  $G$  the subgraph of  $\Gamma$  generated by  $z$ . Then  $G_z$  exactly contains the points  $p$  and  $q$  outside the points in  $G'_p$ , so  $\bar{G}_z$  contains the  $pq$  edge, so this establishes that there is a 1-factor of  $\bar{G}'_p$  and  $(pq)$  is precisely a 1-factor of  $\bar{G}_z$ , so the vertex  $z$  is T-characterized and  $\Gamma_{n,k}$  is not a VP graph.

With this we have proved the theorem.

As an example in figure 12 we can see a  $\Gamma_{8,6}$  graph, which using the symbols from Theorem 9, we indicate (in bold) the  $G_z$  1-factor for the vertex  $z$ .

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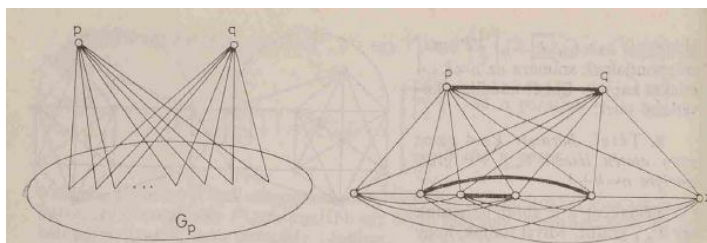


Figure 11

Figure 12

10.Theorem. For any case  $k \geq 2$ , there exist  $\Gamma_{n,k}$   $k$ -regular graphs for which  $n = k + 3$  and  $\Gamma_{n,k}$  does not contain a 1-factor.

PROOF. If  $k \geq 3$ , then we can examine the vertex set  $P$  on  $n$  vertices and select from these three vertices we label  $p_1, p_2$ , and  $p_3$ . Now if we connect  $p_1, p_2$ , and  $p_3$  in order to the remaining vertices of  $P \setminus \{p_1, p_2, p_3\}$  as in figure 13, then the degree of these three vertices will be exactly  $k = n - 3$ , while the degree of every other vertex in  $P \setminus \{p_1, p_2, p_3\}$  is exactly 3. We denote the graph arising from this  $G_1$ .

Now from the vertices in  $P \setminus \{p_1, p_2, p_3\}$  we create an arbitrary regular graph of degree  $k-3$  which we denote as  $G_2$ . If the graphs  $G_1$  and  $G_2$  intersect, we get a  $k = n - 3$  degree regular  $\Gamma_{n,k}$  graph. We create a  $\bar{\Gamma}_{n,k}$  graph. Due to the above construction, the points in  $\{p_1, p_2, p_3\}$  in the general case will not be neighbors to the vertices in  $P \setminus \{p_1, p_2, p_3\}$  of  $\bar{\Gamma}_{n,k}$ . That is,  $\bar{\Gamma}_{n,k}$  splits into two components, for which the portion containing the vertices  $p_1, p_2$ , and  $p_3$  does not contain a 1-factor, independent of whether  $k$  is even or odd. Thus  $\bar{\Gamma}_{n,k}$  does not contain a 1-factor.

If  $k = 2$ , then  $n = 5$ , in which case due to being an odd number  $\Gamma_{n,k}$  does not contain a 1-factor. With this the theorem is proved.

It is easy to see that if  $k = 1$  and  $n = 4$  the theorem does not hold. The well-known Kuratowski graph is achieved in the theorem construction for the case  $k = 3$ , see figure 14.

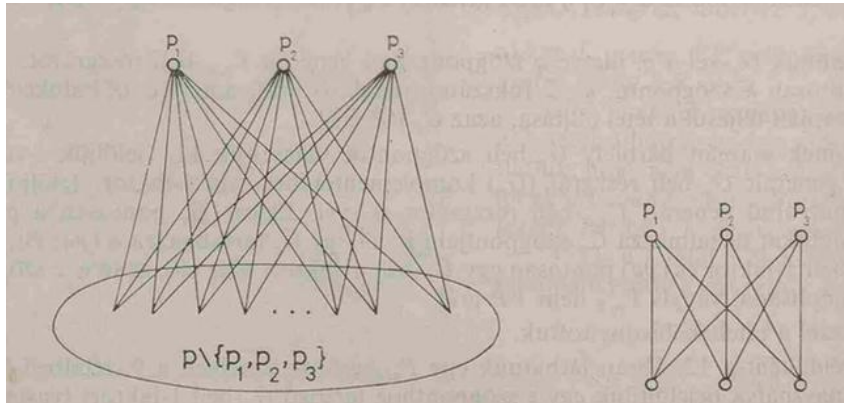


Figure 13

Figure 14

[page 374]

The following theorem gives a method for further specifying the inequality in (25), allowing a specific determination of the value of  $n_{\min}$ .

**Theorem 11.** For any even  $k \geq 6$ , there exists a  $\Gamma \in \mathbf{PR}$  VP graph where  $n = k + 3$ .

**PROOF.** Let  $\Gamma_{n,k}$  be a graph arising from the construction in the proof of Theorem 10 (see the figure in 13) with the restriction (using the symbols from Theorem 10) that the  $G_2$  subgraph should not contain a 1-factor. By Theorem 10 we assume that one such  $G_2$  always exists. We show that every vertex of  $\Gamma_{n,k}$  is T-characterized.

Further suppose  $p$  is an arbitrary element in  $\{p_1, p_2, p_3\}$  and  $q$  an arbitrary vertex of  $G_2$ .

1. The subgraph generated by vertex  $p$  is exactly  $G_2$ , and in addition we assumed that in  $\bar{G}_2$  there is no 1-factor, so  $p$  is not T-characterized.
2. If we denote as  $G_q$  the subgraph generated by vertex  $q$ , then the set  $K$  of vertices of  $G_q$  are  $\{p_1, p_2, p_3\}$  together with the neighbors of  $q$  in  $G_q$ . As each member of  $\{p_1, p_2, p_3\}$  is a neighbor of every vertex in  $K$ ,  $\bar{G}_p$  can be divided into two components, the vertex sets  $\{p_1, p_2, p_3\}$  and  $K$ . The vertices  $\{p_1, p_2, p_3\}$  together with the vertices in  $G$  comprise an odd number of vertices, thus it cannot contain a 1-factor, and thus neither does  $\bar{G}_p$ . Hence  $q$  is clearly not T-characterized, which proves the theorem.

Using Theorems 9 and 11, we still only know an estimated value of  $n_{\min}$  in the inequality in (25) in the case where  $k \geq 6$ . The question is what the situation is when  $k = 2$  and  $k = 4$ ?

In the  $k = 2$  case,  $\Gamma_{n,2} \in \mathbf{PR}_2$  is exactly a Hamiltonian cycle, thus it is simple to see that for no value of  $n$  is  $\Gamma_{n,2}$  a VP graph.

**Theorem 12.** If  $k = 4$  and  $n = k + 3$ , then  $\Gamma \in \mathbf{PR}$  is not a VP graph.

**PROOF.** We select  $\Gamma_{7,4}$  arbitrary  $p$  vertices and examine the subgraph of  $\Gamma_{7,4}$  generated by  $p$ ,  $G_p$ . If  $p$  is not T-characterized, then  $G_p$  contains the representation of the 1-factors of the 4-vertex complete graph constructed from the neighbors of  $p$ . As can be determined from figure 4, this is only possible in two cases: in either the a), d), f), g) case or the b), c), e), h) case.

Using  $q_1$  and  $q_2$  we denote the points of  $\Gamma_{7,4}$  which are not in  $G_p$  and not identical to  $p$  (see figure 15). In the first case  $G_p$  contains the first type of 1-factor representation (see the bold line in figure 16).

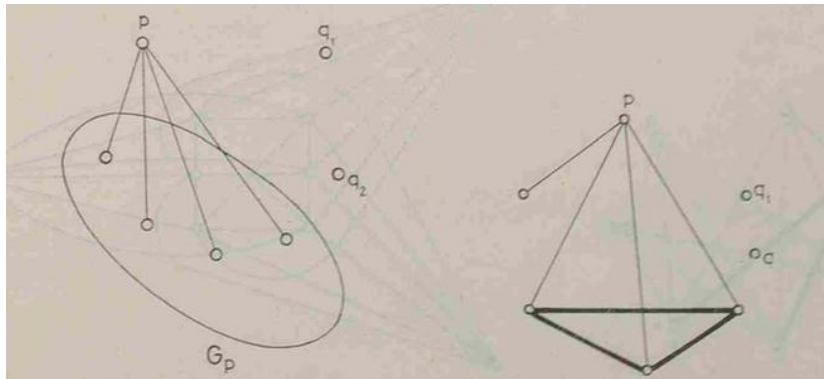


Figure 15

Figure 16

[page 375]

Because  $k = 4$ , vertex  $q_1$  (or  $q_2$ ) is a neighbor with at least two vertices from the representation, so for  $q_2$  (or  $q_1$ ) at most three vertices remain as neighbors. That is, in this case,  $\Gamma_{7,4}$  cannot be constructed (see figure 17).

In the second case  $G_p$  contains the second type of 1-factor representation (shown in bold lines in figure 18).

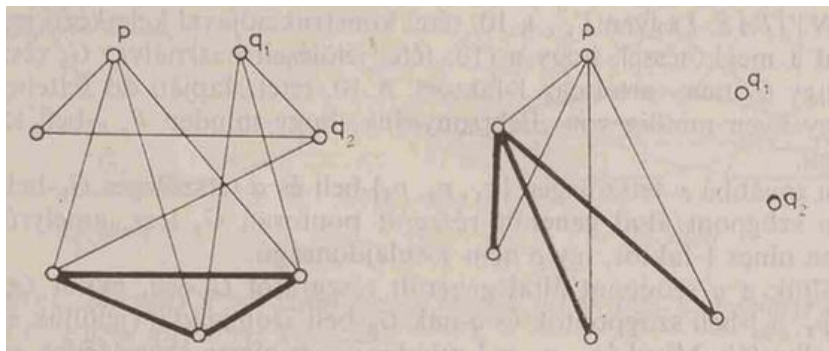


Figure 17

Figure 18

However then only one type of connecting of  $q_1$  and  $q_2$  remains (from the other vertices) for the  $\Gamma_{7,4}$  construction (see figure 19), in which it is easy to find the T-characterized vertex (these are indicated by “T” in figure 19). We remark that using the graph construction from Theorem 10 with much more general assumptions can also give rise to a graph with the desired properties. However, our goal was only to demonstrate existence, for which this proves sufficient.

Theorems 8, 10 and 11 enable to more exactly specify the inequality in (25) and we can give the following theorem:

13. Theorem. For any even number  $k \geq 4$ , the smallest  $n$  for which  $\Gamma \in \mathbf{PR}$  is a VP graph is

$$(26) \quad \begin{aligned} n_{\min} &= k + 3 \text{ if } k \geq 6 \\ n_{\min} &= k + 4 \text{ if } k = 4 \end{aligned}$$

The  $k=4$  case for Theorem 13 is depicted in figure 19 and figure 20 shows an example of the  $k=6$  case.

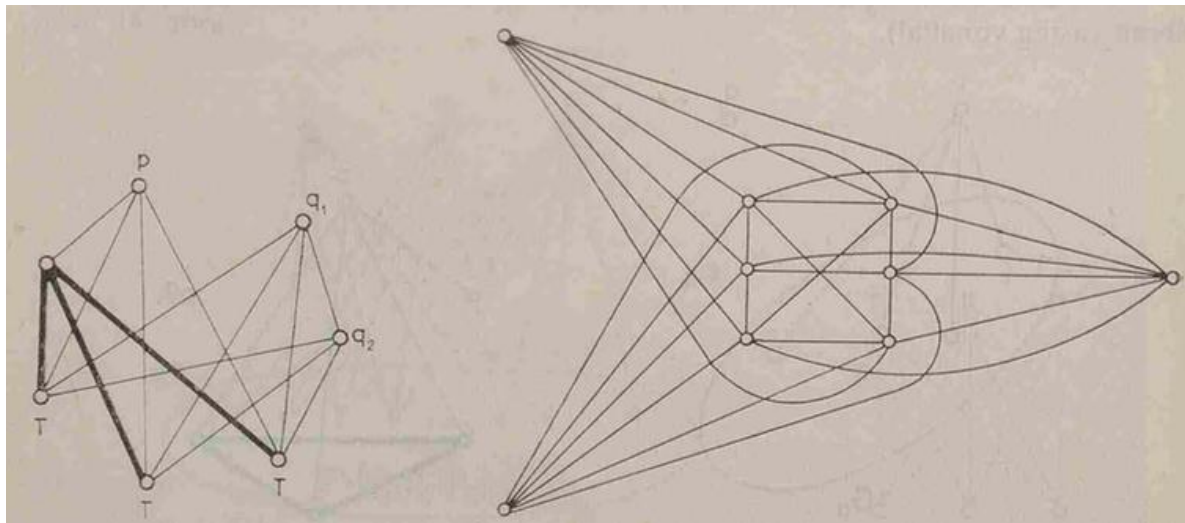


Figure 19

Figure 20

[page 376]

## LITERATURE.

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