

PERFORMANCE OF BLOCK TURBO CODED 16-QAM AND 64-QAM MODULATIONS

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Abstract - In this paper we present the results of our investigation concerning the performance of block turbo codes associated with high spectral efficiency QAM modulations. By using a pragmatic approach, simulation results show that all the block turbo coded QAM modulations we have tested are at 2.9 ± 0.2 dB of their Shannon's limit. The block turbo coded QAM modulations also outperform TCM schemes by at least 1 dB at a BER (Bit Error Rate) of 10^{-5} . Concerning convolutional turbo coded QAM modulations, the block turbo coded QAM modulations exhibit slightly better performance for spectral efficiencies greater than 4 bits/s/Hz.

I. INTRODUCTION

In 1993, C. Berrou [1] showed for the first time that it was possible to realize a digital transmission with a signal to noise ratio (E_b/N_0) close to Shannon's theoretical limit. The transmission scheme proposed by Berrou relies on a concatenated recursive convolutional code associated with an iterative decoding algorithm to protect the binary informations against transmission errors. Although concatenated codes are relatively powerful in terms of minimum distance, one must be very careful when decoding concatenated codes. To obtain near optimum decoding performance, C. Berrou proposed an iterative decoding algorithm based on soft decoding of the component codes and soft decision of the decoded bits. In the case of convolutional codes, soft decoding of the component codes is obtained with the Viterbi [2] algorithm. For the soft decision or reliability of the decoded bits, C. Berrou and P. Adde [3] proposed an efficient and low complexity algorithm based on the work of G. Battail [4]. The correct combination of the reliability information from the demodulator and from the decoder was the key to the success of the "turbo codes."

The success of convolutional turbo codes encouraged us to investigate an equivalent block turbo code. To concatenate the block codes, we used the concept of product codes [5] which was introduced by Elias in 1954. Various iterative decoding algorithms have previously been proposed [6][7] for decoding product codes but the coding gains obtained using these decoding algorithms are well below what one would expect from such powerful codes. The main drawback of these algorithms is that they use algebraic decoders or hard decoders for decoding the component codes. To achieve near optimum decoding of product codes, we must use an iterative decoding algorithm based on soft decoding of the component codes and soft decision of the decoded bits. In 1994 we proposed a new iterative decoding algorithm [8] based on soft decoding of the component codes and soft decision of the decoded bits. We obtained results similar to convolutional

turbo codes and the coding gains were close to the theoretical coding gain expected from product codes when using Maximum Likelihood Sequence Decoding.

In our first paper [8], we tested the performance of block turbo codes associated with a QPSK modulation over a gaussian channel. We showed that for a BER (Bit Error Rate) of 10^{-5} , the signal to noise ratio (E_b/N_0) was at 2.5 dB of Shannon's theoretical limit for block turbo codes with any code rate less or equal to 0.9. Although it is well known that block codes exhibit high code rates, the spectral efficiency of the block turbo coded QPSK modulations will never exceed 2 bits/s/Hz. As modern applications require high spectral efficiency transmission systems, we have investigated the performance of block turbo coded 16-QAM and 64-QAM modulations over a gaussian channel.

This paper describes a method for associating block turbo codes with 16-QAM and 64-QAM modulations and presents the simulation results we have obtained. In section II we recall the properties of product codes and we describe the method we used to associate product codes with QAM modulations. In section III, we give a brief description of the soft decoding algorithm and the soft decision algorithm and also the iterative decoding procedure for decoding block turbo coded QAM modulations. Section IV is dedicated to the performance of block turbo coded QAM modulations. We compare our results with the best known coding schemes which are TCM (Treillis Coded Modulations) and high spectral efficiency convolutional turbo codes. Finally we draw some conclusions in section V.

II. BLOCK TURBO CODED QAM MODULATIONS

The concept of product code is a simple and relatively efficient method to construct powerful codes (that is having large minimum Hamming distance δ) by using classic linear block codes [5]. Let us consider two systematic linear block codes \mathcal{C}^1 having parameters (n_1, k_1, δ_1) and \mathcal{C}^2 having parameters (n_2, k_2, δ_2) where n_i, k_i and δ_i ($i=1,2$) stand for code length, number of information bits and minimum Hamming distance respectively. The product code $\mathcal{P} = \mathcal{C}^1 \otimes \mathcal{C}^2$ is obtained by :

- 1) placing $(k_1 \times k_2)$ information bits in a matrix,
- 2) coding the k_1 rows using code \mathcal{C}^2 ,
- 3) coding the n_2 columns using code \mathcal{C}^1 ,

as illustrated in Fig. 1.

It is shown [5] that all the n_1 rows of the matrix in Fig. 1 are code words of \mathcal{C}^2 exactly as the n_2 columns are code words of \mathcal{C}^1 by construction. Furthermore the

parameters of the resulting product code \mathcal{P} is given by $n=n_1 \times n_2$, $k=k_1 \times k_2$ and $\delta=\delta_1 \times \delta_2$ and the code rate R is given by $R=R_1 \times R_2$ where R_i is the code rate of code \mathcal{C}^i

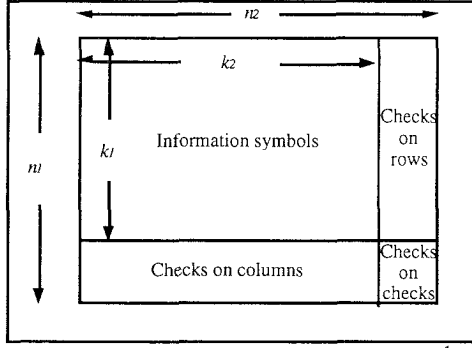


Figure 1 : Construction of product code $\mathcal{P} = \mathcal{C}^1 \otimes \mathcal{C}^2$

In the case of block turbo coded QPSK modulations, the association of a product code with a QPSK modulation was straightforward. Each binary element of the product code was used to select a symbol from the in-phase channel (or from the quadrature channel) of the QPSK constellation. This is not the case for a 16-QAM modulation since each symbol of the in-phase (or quadrature) channel is selected by two binary elements. We used the pragmatic approach proposed by Viterbi [9] to associate the block turbo codes with the QAM modulations which is a very good compromise between decoding complexity and performance. We begin by grouping the binary elements of the coded matrix by two and we shall denote the two binary elements x and y respectively. The binary couple (x,y) then selects a symbol a from the in-phase channel (or quadrature channel) of the 16-QAM constellation as illustrated in Fig. 2.

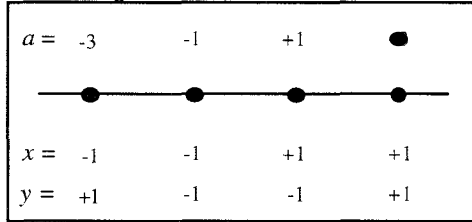


Figure 2 : Binary elements associated to the symbols of the in-phase channel of the 16-QAM modulation.

Symbol a is then transmitted to the receiver. The sample r at the output of the coherent demodulator corresponding to the transmission of symbol a over a gaussian channel is given by relation $r = a + b$ where b is additive white gaussian noise (AWGN). In our previous paper [8] we showed that we require the Log-Likelihood-Ratio (LLR) associated to each binary element of the coded matrix to perform near optimum decoding of product codes. For a 16-QAM modulation, the LLR of x is defined by the following relation :

$$(LLRx) = \ln \left(\frac{\Pr\{x = +1 / r\}}{\Pr\{x = -1 / r\}} \right) \quad (1)$$

By using Bayes rule and assuming that the transmitted symbols a are identically distributed over the four values $\{\pm 1, \pm 3\}$, relation (1) yields the following expression :

$$(LLRx) = \ln \left(\frac{p\{r/a = +1\} + p\{r/a = +3\}}{p\{r/a = -1\} + p\{r/a = -3\}} \right) \quad (2)$$

where $p\{r/a\}$ is the probability density function of r conditionally to transmitted symbol a . At high signal to noise ratio, it is easily shown that relation (2) can be approximated by :

$$(LLRx) = \begin{cases} r & \text{if } |r| \leq 2 \\ 2(r-1) & \text{if } r > +2 \\ 2(r+1) & \text{if } r < -2 \end{cases} \quad (3)$$

where the LLR has been normalized with respect to the standard deviation of r . By using a similar demonstration we can show that at high signal to noise ratio, the normalized LLR of y can be approximated by :

$$(LLRy) = \begin{cases} +(r-2) & \text{if } |r| \geq 0 \\ -(r+2) & \text{if } |r| < 0 \end{cases} \quad (4)$$

On receiving samples $\{\mathbf{R}\}$ corresponding to a coded matrix, we compute the LLR of the binary elements of the matrix according to relations (3) and (4). $(LLRx)$ and $(LLRy)$ are two random variables with probability density functions which have a gaussian distribution. However the parameters of the two gaussian distributions are different. In order to reduce the complexity of the decoder, we distributed the binary elements x and y in the coded matrix so as to prevent the accumulation of binary elements x or y along the columns or rows of the coded matrix. Having derived the LLR of the binary elements of the coded matrix we shall now describe the iterative decoding algorithm.

III. OPTIMUM DECODING OF PRODUCT CODES

To simplify our notations, the LLR of the binary elements corresponding to a row or column of the coded matrix will be noted $\mathbf{R} = (r_1, \dots, r_j, \dots, r_n)$. Optimum maximum likelihood sequence decoding of \mathbf{R} is given by :

$$\mathbf{D} = \mathbf{C}^i \text{ if } |\mathbf{R} - \mathbf{C}^i|^2 < |\mathbf{R} - \mathbf{C}^l|^2 \quad \forall l \in [1, 2^k], l \neq i \quad (5)$$

where $\mathbf{C}^i = (c_1^i, \dots, c_j^i, \dots, c_n^i)$ is the i^{th} code word of the component code \mathcal{C} with parameters (n, k, δ) . Decision $\mathbf{D} = (d_1, \dots, d_j, \dots, d_n)$ corresponds to the maximum likelihood transmitted sequence conditionally to \mathbf{R} and :

$$|\mathbf{R} - \mathbf{C}^i|^2 = \sum_{j=1}^n (r_j - c_j^i)^2 \quad (6)$$

is the squared Euclidean distance between \mathbf{R} and \mathbf{C}^i . For block codes with a high code rate R , the number of code words 2^k is relatively large and maximum likelihood sequence decoding is too complex for implementation. To reduce the complexity of the soft decoder we used the Chase algorithm [10] which approximates maximum likelihood sequence decoding of block codes with a low computation complexity and a small performance degradation. Instead of reviewing all the code words $l = 1..2^k$, the Chase algorithm searches for the code words at Hamming distance within a sphere of radius $(\delta-1)$ centered on $\mathbf{Y} = (y_1, \dots, y_j, \dots, y_n)$ where y_j is the sign of r_j (± 1). To further reduce the number of reviewed code words, only the most probable code words within the sphere are selected by using channel information

R. This search procedure can be decomposed in three steps :

- 1) determine the position of the $[\delta/2]$ least reliable binary elements of \mathbf{Y} using \mathbf{R} ,
- 2) form test patterns \mathbf{T}^q defined as all the n -dimensional binary vectors with a "1" in the $[\delta/2]$ least reliable positions and "0" in the other positions, two "1" in the $[\delta/2]$ least reliable positions and "0" in the other positions, ..., $[\delta/2]$ "1" in the $[\delta/2]$ least reliable positions and "0" in the other positions,
- 3) decode $\mathbf{Z}^q = \mathbf{Y} \oplus \mathbf{T}^q$ using an algebraic decoder and memorize the code words \mathbf{C}^q .

Maximum likelihood transmitted sequence \mathbf{D} is then given by decision rule (5) with the reviewed code words restricted to those found at step 3 above. The Chase algorithm yields the maximum likelihood sequence \mathbf{D} for a given soft input \mathbf{R} . We must now compute the reliability of the decoded bits corresponding to the maximum likelihood sequence \mathbf{D} . This reliability function is given by the Log Likelihood Ratio (LLR) of the decision d_j which is defined by :

$$(LLR)_j = \ln \left(\frac{\Pr\{e_j = +1 / \mathbf{R}\}}{\Pr\{e_j = -1 / \mathbf{R}\}} \right) \quad (7)$$

where e_j is the binary element in position j of transmitted code word \mathbf{E} . By considering the different authorized code words, the numerator of (7) can be expressed as :

$$\Pr\{e_j = +1 / \mathbf{R}\} = \sum_{\mathbf{C}^i \in S_j^{+1}} \Pr\{\mathbf{E} = \mathbf{C}^i / \mathbf{R}\} \quad (8)$$

where S_j^{+1} is the set of code words such that $c_j^i = +1$ and the denominator of (7) can be expressed as :

$$\Pr\{e_j = -1 / \mathbf{R}\} = \sum_{\mathbf{C}^i \in S_j^{-1}} \Pr\{\mathbf{E} = \mathbf{C}^i / \mathbf{R}\} \quad (9)$$

where S_j^{-1} is the set of code words such that $c_j^i = -1$. By applying BAYES rule to (8) and (9) and assuming that the different code words are uniformly distributed, we obtain for the LLR of d_j the following expression :

$$(LLR)_j = \ln \left(\frac{\sum_{i \in S_j^{+1}} p\{\mathbf{R} / \mathbf{E} = \mathbf{C}^i\}}{\sum_{i \in S_j^{-1}} p\{\mathbf{R} / \mathbf{E} = \mathbf{C}^i\}} \right) \quad (10)$$

where $p\{\}$ is the probability density function of \mathbf{R} conditionally to \mathbf{E} . As shown in [8], relation (10) can be approximated by :

$$(LLR)_j \approx \frac{1}{2\sigma^2} \left(\left| \mathbf{R} - \mathbf{C}^{-1(j)} \right|^2 - \left| \mathbf{R} - \mathbf{C}^{+1(j)} \right|^2 \right) \quad (11)$$

where $\mathbf{C}^{+1(j)}$ is the code word in S_j^{+1} at minimum distance from \mathbf{R} and $\mathbf{C}^{-1(j)}$ is the code word in S_j^{-1} at minimum distance from \mathbf{R} . σ is the standard deviation of the components in vector \mathbf{R} . By expanding relation (11) using (6) we finally obtain the following expression :

$$(LLR)_j \approx \frac{2}{\sigma^2} \left(r_j + \sum_{l=1, l \neq j}^n \eta c_l^{+1(j)} p_l \right) \quad (12)$$

$$\text{where : } p_l = \begin{cases} 0 & \text{if } c_l^{+1(j)} = c_l^{-1(j)} \\ 1 & \text{if } c_l^{+1(j)} \neq c_l^{-1(j)} \end{cases} \quad (13)$$

The normalization of the approximated LLR of d_j with respect to $2/\sigma^2$ yields the new variable $r'_j = r_j + w_j$ where :

$$w_j = \sum_{l=1, l \neq j}^n \eta c_l^{+1(j)} p_l \quad (14)$$

The estimated normalized LLR of decision d_j , r'_j is given by r_j plus w_j which is a function of the code words at minimum Euclidean distance from \mathbf{R} and $\{r_l\}$ with $l \neq j$. The term w_j is the additional information given by the decoder to the reliability of the decoded bit. Since w_j is a combination of random variables r_l , it is also a random variable with a gaussian distribution. The approximated normalized LLR of d_j (r'_j) is an estimation of the soft decision of the block decoder. It has the same sign as d_j and its absolute value indicates the reliability of the decision.

To compute the normalized LLR using (12) we suppose that we have identified the two code words $\mathbf{C}^{+1(j)}$ and $\mathbf{C}^{-1(j)}$. This can be done by increasing the number of least reliable bits and test patterns in the Chase algorithm (see section II). For each code word \mathbf{C}^q found at step 3, we compute its Euclidean distance from \mathbf{R} :

$$M^q = \left| \mathbf{R} - \mathbf{C}^q \right|^2 \quad (15)$$

As in the Chase algorithm we select the code word \mathbf{D} at minimum Euclidean distance $M^{\mathbf{D}}$ from \mathbf{R} . Then we search for the code word \mathbf{C} at minimum Euclidean distance $M^{\mathbf{C}}$ from \mathbf{R} such that $c_j \neq d_j$. If we find such a code word then we compute the soft decision for bit d_j using the relation given below :

$$r'_j = \left(\frac{M^{\mathbf{C}} - M^{\mathbf{D}}}{4} \right) d_j \quad (16)$$

else we use the relation :

$$r'_j = r_j + (\beta \times d_j) \quad (17)$$

where β is a constant which is a function of the BER (Bit Error Rate) and is optimized by simulation. Equation (17) is justified by the fact that w_j has the same sign as the LLR of d_j but we have not been able to compute the amplitude of the reliability since we have not found code word \mathbf{C} . We now have all the elements to perform the iterative decoding of block turbo coded QAM modulations.

On receiving the matrix $[\mathbf{R}]$ corresponding to the LLR of the binary elements in the coded matrix, the first decoder performs the soft decoding of the rows (or columns) of the matrix, estimates the normalized LLR $[\mathbf{R}']$ using either relation (16) or (17) and gives as output $[\mathbf{W}(1)]$. Then the next decoder performs the same operations on the columns (or rows) using as input data $[\mathbf{R}(1)] = [\mathbf{R}] + \alpha(1)[\mathbf{W}(1)]$. This decoding procedure is then iterated by cascading the

elementary decoders illustrated in Fig. 3. The coefficient $\alpha(k)$ is used to reduce the influence of $[W(k)]$ in the first iterations when the BER is relatively high and thus $[W(k)]$ is not absolutely reliable.

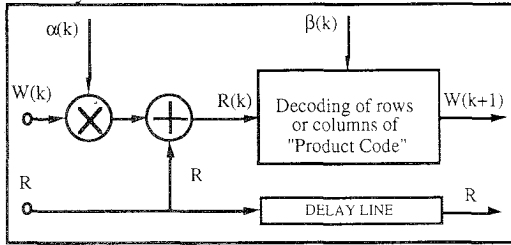


Figure 3 : Block diagram of elementary decoder.

IV. PERFORMANCE OF BLOCK TURBO CODED QAM MODULATIONS

A close analysis of the iterative decoder shows that the parameters $\alpha(k)$ and $\beta(k)$ are not independent and that the optimum value for these parameters depends on the product code used. This implies an optimization of the parameters for each product code. To overcome this problem, we normalized the average value of the computed w_j , which is a random variable with a gaussian distribution, to +1 for samples $w_j > 0$ and -1 for samples $w_j < 0$. By doing so we have noticed that the optimum value for parameters α and β are practically independent and are also independent of the product code. This simplified the task in evaluating the performance of the different product codes. We used the following values for α and β : $\alpha = [0, 0.5, 0.7, 0.9, 1, 1, 1, 1]$ and $\beta = [0.2, 0.3, 0.5, 0.7, 0.9, 1, 1, 1]$.

It is clear that the performance of the turbo decoder depends on the number of test patterns. In our investigation, we used seventeen test patterns based on the five least reliable bits. This is a compromise between performance and complexity.

We shall now present the simulated performance of block turbo coded QAM modulations transmitted over a gaussian channel. We evaluated the performance of different block turbo codes based on extended BCH codes associated with 16-QAM and 64-QAM modulations. The BER (Bit Error Rate) was evaluated using Monte Carlo simulations for different values of the signal to noise ratio (E_b/N_0). The BER is given for one hundred matrices having at least one bit false after decoding down to a BER of 10^{-6} . We used identical BCH codes for coding the rows and columns of the product codes, that is $\mathcal{C}^1 = \mathcal{C}^2$.

The simulation results show that after iteration-4, the amelioration of the coding gain becomes negligible [8] because of the steep slope of the BER curve (see Fig. 4). The signal to noise ratio for a BER of 10^{-5} at iteration-4 is at 2.8 dB of Shannon's theoretical limit. In Fig. 5 we have plotted the BER versus signal to noise ratio of different block turbo coded 16-QAM modulations at iteration-4. As a reference, we have also plotted the BER versus signal to noise ratio of the 64-state 16-QAM TCM with a spectral efficiency of 3 bits/s/Hz. From Fig. 5 we note that the slopes of all the BER curves of the block turbo coded 16-QAM modulations are steeper than that of the 64-state TCM. Furthermore, for a BER less than 10^{-6} , the block turbo coded 16-QAM modulations outperform the 64-state TCM.

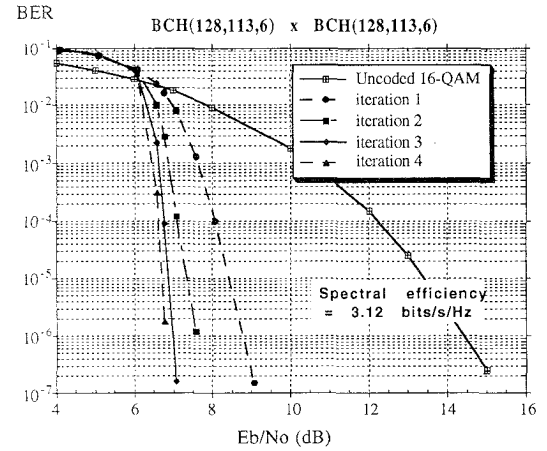


Figure 4 : Bit Error Rate versus SNR (E_b/N_0) for the first four iterations when using a 16-QAM modulation.

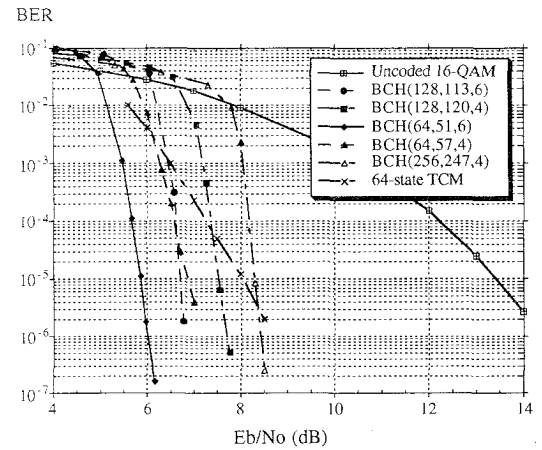


Figure 5 : BER versus signal to noise ratio at iteration-4 for different block turbo coded 16-QAM modulations.

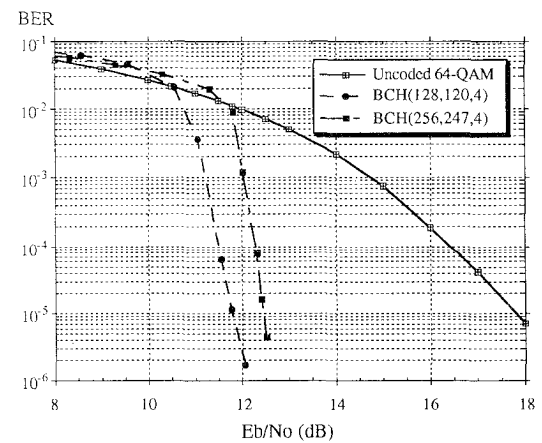


Figure 6 : BER versus signal to noise ratio at iteration-4 for different block turbo coded 64-QAM modulations.

In Fig. 6 we have plotted the BER versus signal to noise ratio of two block turbo coded 64-QAM modulations at iteration-4. As mentioned in our previous paper [8], we observe from Fig. 5 and Fig. 6 that the slope of the BER curve increases with the size of the matrix and with the minimum Hamming distance of the product code. We also notice that all the block turbo coded QAM modulations we have tested are at 2.9 ± 0.2 dB of Shannon's theoretical limit. This is slightly greater than what we obtained previously [8] with block turbo coded QPSK modulations.

In order to compare the different coding schemes, we have plotted in Fig. 7 the signal to noise ratio of different coding schemes at a BER of 10^{-5} as a function of their spectral efficiency. Concerning convolutional and block turbo-codes, results at the third iteration are presented. This comparison clearly shows that block turbo codes outperform the 64-state TCM schemes by at least 1 dB. We also notice a slight advantage for block turbo codes over convolutional turbo codes [11] for spectral efficiencies greater than 4 bits/s/Hz. This is due to the fact that there is a degradation of the performance of convolutional turbo codes when the code is punctured so as to increase its code rate [11].

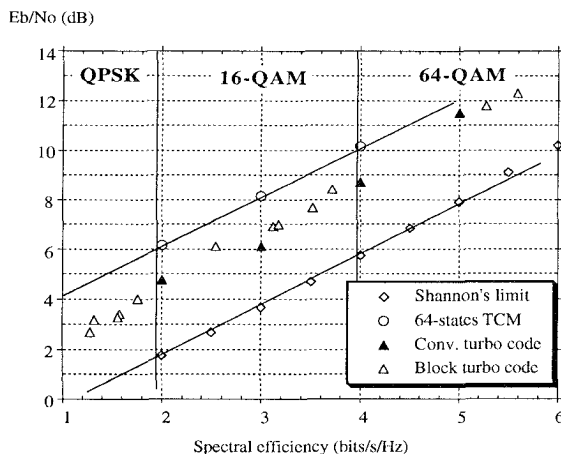


Figure 7 : (E_b/N_0) at a BER of 10^{-5} versus spectral efficiency for different coding schemes.

V. CONCLUSION

In this paper we have presented a new coding scheme which is based on block turbo codes associated with QAM modulations. By using a pragmatic approach, we have derived a very simple method to perform the iterative decoding of block turbo coded QAM modulations and we have simulated the performance of different block turbo coded QAM modulations. We have noticed that, for all the block turbo coded QAM modulations we have tested, the signal to noise ratio at a BER of 10^{-5} is at 2.9 ± 0.2 dB of their respective Shannon's limit. These block turbo coded QAM modulations exhibit much better performance than the 64-states TCM coding schemes. Furthermore, for spectral efficiencies greater than 4 bits/s/Hz, the block turbo coded QAM modulations show better performance than convolutional turbo coded QAM modulations. This clearly

shows that block turbo codes are actually the most efficient coding schemes for digital transmission systems requiring a high spectral efficiency.

However, the main drawback of turbo codes in general is the delay introduced by the iterative decoder which can be critical for some applications. In the case of block turbo codes, this delay can be reduced by using parallel decoding of the rows and columns of the coded matrix and also by reducing the complexity of the iterative decoding process. Furthermore, preliminary results show that block turbo codes exhibit no performance degradation when using a 4-bit quantization of the input data while convolutional turbo codes exhibit a minimum degradation of 0.5 dB. In our opinion, although simulation results show a slight advantage of convolutional turbo codes for code rates less than 0.75, we believe that this threshold will be much smaller when comparing circuit performance.

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