

Spherical Bessel functions as fourier transforms of Welch tapers

Nick Stokes

Feb 24 2015

1 Introduction

Define

$$P_n(\omega) = \int_{-1}^1 t(1-t^2)^n \sin(\omega t) dt$$

This is the Fourier transform of the trend operator with n iterations of Welch tapering. A complementary integral is

$$Q_n(\omega) = \int_{-1}^1 (1-t^2)^n \cos(\omega t) dt$$

Differentiate each (through the integral sign)

$$Q'_n = -P_n$$

$$P'_n = Q_n - Q_{n+1}$$

Now make an integration by parts expression and then eliminate Q to get ode for P:

$$\int_{-1}^1 ((1-t^2)^n \sin(\omega t))' dt = 0 = -2nP_{n-1} + \omega Q_n, n > 0$$

$$P_n'' = Q_n' - Q_{n+1}' = -P_n - 2(n+1)/(P_n/\omega)'$$

$$\omega^2 P_n'' + 2(n+1)\omega P_n' + (\omega^2 - 2(n+1))P_n = 0$$

Now Bessel's equation for $J_m(w)$ is

$$w^2 y'' + w y' + (w^2 - m^2)y = 0$$

Set

$$y = w^{m-1} z$$

$$w^2 z'' + 2(m-1)w z' + (m-2)(m-1)z + w z' + (m-1)z + (w^2 - m^2)z = 0$$

$$w^2 z'' + (2m-1)w z' + (w^2 + 1 - 2m)z = 0$$

So P_n satisfies Bessels equation for $w^{1-m} J_m(t)$, $m = n + 3/2$

It remains to normalize so that the Bessel solution has slope 1

$$J_\nu(w) = \frac{(w/2)^\nu}{\Gamma(\nu+1)}(1 + \dots)$$

So the sequence of Fourier transforms of the successive smoothed differentiators is:

$$(2/\omega)^{n+1/2} \Gamma(n+3/2) \omega J_{n+1/2}(\omega), n = 1, 2, 3, \dots$$